Chapter 14

14.1 (a) 2-band polyphase decomposition – Using \( P(z) = 1 - cz^{-1} \), we express

\[
H(z) = \frac{a + bz^{-1}}{1 + cz^{-1}}
\]

as

\[
H(z) = \frac{(a + bz^{-1})(1 - cz^{-1})}{(1 + cz^{-1})(1 - cz^{-1})} = \frac{a + (b - ac)z^{-1} - bcz^{-2}}{1 - c^2z^{-2}}
\]

\[
= \left( \frac{a - bcz^{-2}}{1 - c^2z^{-2}} \right) + z^{-1} \left( \frac{b - ac}{1 - c^2z^{-2}} \right).
\]

Hence, \( E_0(z^2) = \frac{a - bcz^{-2}}{1 - c^2z^{-2}} \) and

\[
E_1(z^2) = \frac{b - ac}{1 - c^2z^{-2}}.
\]

(b) 3-band polyphase decomposition – Using \( P(z) = 1 - cz^{-1} + c^2z^{-2} \), we rewrite \( H(z) \) as

\[
H(z) = \frac{(a + bz^{-1})(1 - cz^{-1} + c^2z^{-2})}{(1 + cz^{-1})(1 - cz^{-1} + c^2z^{-2})} = \frac{a + (b - ac)z^{-1} + c(ac - b)z^{-2} + bc^2z^{-3}}{1 - c^3z^{-3}}
\]

\[
= \left( \frac{a + bc^2z^{-3}}{1 - c^3z^{-3}} \right) + z^{-1} \left( \frac{b - ac}{1 - c^3z^{-3}} \right) + z^{-2} \left( \frac{c(ac - b)}{1 - c^3z^{-3}} \right)
\]

Hence, \( E_0(z^3) = \frac{a + bc^2z^{-3}}{1 - c^3z^{-3}} \), \( E_1(z^3) = \frac{b - ac}{1 - c^3z^{-3}} \), \( E_2(z^3) = \frac{c(ac - b)}{1 - c^3z^{-3}} \).

14.2 (a) \( H_a(z) = \frac{a + bz^{-1}}{1 + cz^{-1}} \), \( H_a(zW_2^1) = H_a(-z) = \frac{a - bz^{-1}}{1 - cz^{-1}} \). Thus,

\[
E_0(z^2) = \frac{1}{2} [H_a(z) + H_a(-z)] = \frac{1}{2} \left( \frac{a + bz^{-1}}{1 + cz^{-1}} + \frac{a - bz^{-1}}{1 - cz^{-1}} \right) = \frac{a - bcz^{-2}}{1 - c^2z^{-2}}
\]

and

\[
z^{-1}E_1(z^2) = \frac{1}{2} [H_a(z) - H_a(-z)] = \frac{1}{2} \left( \frac{a + bz^{-1}}{1 + cz^{-1}} - \frac{a - bz^{-1}}{1 - cz^{-1}} \right) = \frac{(b - ac)z^{-1}}{1 - c^2z^{-2}}.
\]

Hence, a 2-band polyphase decomposition of \( H_a(z) \) is given by

\[
H_a(z) = \left( \frac{a - bcz^{-2}}{1 - c^2z^{-2}} \right) + z^{-1} \left( \frac{b - ac}{1 - c^2z^{-2}} \right).
\]

(b) \( H_b(z) = \frac{3 - 4z^{-1} + 2.1z^{-2}}{1 - 0.8z^{-1} + 0.7z^{-2}} \), \( H_b(zW_2^1) = H_b(-z) = \frac{3 + 4z^{-1} + 2.1z^{-2}}{1 + 0.8z^{-1} + 0.7z^{-2}} \).

Thus,

\[
E_0(z^2) = \frac{1}{2} [H_b(z) + H_b(-z)] = \frac{1}{2} \left( \frac{3 - 4z^{-1} + 2.1z^{-2}}{1 - 0.8z^{-1} + 0.7z^{-2}} + \frac{3 + 4z^{-1} + 2.1z^{-2}}{1 + 0.8z^{-1} + 0.7z^{-2}} \right)
\]
\[
\frac{3 + z^{-2} + 1.47z^{-4}}{1 + 0.76z^{-2} + 0.49z^{-4}}.
\]

\[
z^{-1}E_1(z^2) = \frac{1}{2} [H_b(z) - H_b(-z)] = \frac{1}{2} \left( \frac{3 - 4z^{-1} + 2.1z^{-2}}{1 - 0.8z^{-1} + 0.7z^{-2}} - \frac{3 + 4z^{-1} + 2.1z^{-2}}{1 + 0.8z^{-1} + 0.7z^{-2}} \right)
\]

\[
= -1.6z^{-1} - 1.12z^{-3}
\]

\[
\frac{1}{1 + 0.76z^{-2} + 0.49z^{-4}}.
\]

Hence, \(E_1(z^2) = \frac{-1.6 - 1.12z^{-2}}{1 + 0.76z^{-2} + 0.49z^{-4}}.\)

Hence, a 2-band polyphase decomposition of \(H_b(z)\) is given by

\[
H_b(z) = \left( \frac{3 + z^{-2} + 1.47z^{-4}}{1 + 0.76z^{-2} + 0.49z^{-4}} \right) + z^{-1} \left( \frac{-1.6 - 1.12z^{-2}}{1 + 0.76z^{-2} + 0.49z^{-4}} \right).
\]

(c) \(H_c(z) = \frac{4 + 2.5z^{-1} - 3.5z^{-2} + 2z^{-3}}{1 - 0.4z^{-1} + 0.78z^{-2} + 0.18z^{-3}}.\)

\(H_c(-z) = \frac{4 - 2.5z^{-1} - 3.5z^{-2} - 2z^{-3}}{1 + 0.4z^{-1} + 0.78z^{-2} - 0.18z^{-3}}.\)

\[
E_0(z^2) = \frac{1}{2} [H_c(z) + H_c(-z)]
\]

\[
= \frac{1}{2} \left( \frac{4 + 2.5z^{-1} - 3.5z^{-2} + 2z^{-3}}{1 - 0.4z^{-1} + 0.78z^{-2} + 0.18z^{-3}} + \frac{4 - 2.5z^{-1} - 3.5z^{-2} - 2z^{-3}}{1 + 0.4z^{-1} + 0.78z^{-2} - 0.18z^{-3}} \right)
\]

\[
= \frac{4 + 0.62z^{-2} - 2.38z^{-4} - 0.36z^{-6}}{1 + 1.4z^{-2} + 0.7524z^{-4} - 0.0324z^{-6}}.
\]

\[
z^{-1}E_1(z^2) = \frac{1}{2} [H_c(z) - H_c(-z)] = \frac{1}{2} \left( \frac{4 + 2.5z^{-1} - 3.5z^{-2} + 2z^{-3}}{1 - 0.4z^{-1} + 0.78z^{-2} + 0.18z^{-3}} - \frac{4 - 2.5z^{-1} - 3.5z^{-2} - 2z^{-3}}{1 + 0.4z^{-1} + 0.78z^{-2} - 0.18z^{-3}} \right)
\]

\[
= \frac{4.1z^{-1} + 1.83z^{-3} + 2.19z^{-5}}{1 + 1.4z^{-2} + 0.7524z^{-4} - 0.0324z^{-6}}.\]

Thus,

\[
E_1(z^2) = \frac{4.1 + 1.83z^{-2} + 2.19z^{-4}}{1 + 1.4z^{-2} + 0.7524z^{-4} - 0.0324z^{-6}}.
\]

Hence, a 2-band polyphase decomposition of \(H_c(z)\) is given by

\[
H_c(z) = \left( \frac{4 + 0.62z^{-2} - 2.38z^{-4} - 0.36z^{-6}}{1 + 1.4z^{-2} + 0.7524z^{-4} - 0.0324z^{-6}} \right) + z^{-1} \left( \frac{4.1 + 1.83z^{-2} + 2.19z^{-4}}{1 + 1.4z^{-2} + 0.7524z^{-4} - 0.0324z^{-6}} \right).
\]

14.3 (a) \(H_1(z) = \frac{3 - 4z^{-1}}{1 - 0.5z^{-1}} = \sum_{k=0}^{2} z^{-k} E_k(z^3).\) We can write,
\[
\begin{bmatrix}
H_1(z) \\
H_1(W_3^1 z) \\
H_1(W_3^2 z)
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
1 & W_3^{-1} & W_3^{-2} \\
1 & W_3^{-2} & W_3^{-1}
\end{bmatrix} \begin{bmatrix}
E_0(z^3) \\
z^{-1}E_1(z^3) \\
z^{-2}E_2(z^3)
\end{bmatrix}.
\]
Thus,
\[
\begin{bmatrix}
E_0(z^3) \\
z^{-1}E_1(z^3) \\
z^{-2}E_2(z^3)
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
1 & W_3^{-1} & W_3^{-2} \\
1 & W_3^{-2} & W_3^{-1}
\end{bmatrix}^{-1} \begin{bmatrix}
H_1(z) \\
H_1(W_3^1 z) \\
H_1(W_3^2 z)
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
1 & 1 & 1 \\
1 & W_3^1 & W_3^2 \\
1 & W_3^2 & W_3^1
\end{bmatrix} \begin{bmatrix}
H_1(z) \\
H_1(W_3^1 z) \\
H_1(W_3^2 z)
\end{bmatrix}.
\]

Therefore,
\[
E_0(z^3) = \frac{1}{3} [H_1(z) + H_1(zW_3^1) + H_1(zW_3^2)]
\]
\[
= \frac{1}{3} \left( \frac{3 - 4z^{-1}}{1 - 0.5z^{-1}} + \frac{3 - 4e^{j2\pi/3}z^{-1}}{1 - 0.5e^{j2\pi/3}z^{-1}} + \frac{3 - 4e^{j4\pi/3}z^{-1}}{1 - 0.5e^{j4\pi/3}z^{-1}} \right) = \frac{3 - z^{-3}}{1 - (0.5)^3z^{-3}},
\]
\[
z^{-1}E_1(z^3) = \frac{1}{3} [H_1(z) + W_3^1 H_1(zW_3^1) + W_3^2 H_1(zW_3^2)]
\]
\[
= \frac{1}{3} \left( \frac{3 - 4z^{-1}}{1 - 0.5z^{-1}} + 3 - 4e^{j2\pi/3}z^{-1} + e^{j4\pi/3} \frac{3 - 4e^{j4\pi/3}z^{-1}}{1 - 0.5e^{j4\pi/3}z^{-1}} \right)
\]
\[
= z^{-1} \left( \frac{-2.5}{1 - (0.5)^3z^{-3}} \right),
\]
\[
z^{-2}E_2(z^3) = \frac{1}{3} [H_1(z) + W_3^2 H_1(zW_3^1) + W_3^1 H_1(zW_3^2)]
\]
\[
= \frac{1}{3} \left( \frac{3 - 4z^{-1}}{1 - 0.5z^{-1}} + e^{j4\pi/3} \frac{3 - 4e^{j2\pi/3}z^{-1}}{1 - 0.5e^{j2\pi/3}z^{-1}} + e^{j2\pi/3} \frac{3 - 4e^{j4\pi/3}z^{-1}}{1 - 0.5e^{j4\pi/3}z^{-1}} \right)
\]
\[
= z^{-2} \left( \frac{-1.25}{1 - (0.5)^3z^{-3}} \right).
\]

Hence, 
\[
E_0(z) = \frac{3 - z^{-1}}{1 - 0.125z^{-1}}, \quad E_1(z) = \frac{-2.5}{1 - 0.125z^{-1}}, \quad E_2(z) = \frac{-1.25}{1 - 0.125z^{-1}}.
\]

(b) \[
H_2(z) = \frac{4 + 2.1z^{-1} - 3.4z^{-2}}{1 - 0.8z^{-1} + 0.6z^{-2}} = \sum_{k=0}^{2} z^{-k} E_k(z^3).
\]

We can write,
\[
\begin{bmatrix}
H_2(z) \\
H_2(W_3^1 z) \\
H_2(W_3^2 z)
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
1 & W_3^{-1} & W_3^{-2} \\
1 & W_3^{-2} & W_3^{-1}
\end{bmatrix} \begin{bmatrix}
E_0(z^3) \\
z^{-1}E_1(z^3) \\
z^{-2}E_2(z^3)
\end{bmatrix}.
\]
Thus,
\[
\begin{bmatrix}
E_0(z^3) \\
z^{-1}E_1(z^3) \\
z^{-2}E_2(z^3)
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 1 \\
1 & W_3^{-1} & W_3^{-2} \\
1 & W_3^{-2} & W_3^{-1}
\end{bmatrix}^{-1} \begin{bmatrix}
H_2(z) \\
H_2(W_3^1 z) \\
H_2(W_3^2 z)
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
1 & 1 & 1 \\
1 & W_3^1 & W_3^2 \\
1 & W_3^2 & W_3^1
\end{bmatrix} \begin{bmatrix}
H_2(z) \\
H_2(W_3^1 z) \\
H_2(W_3^2 z)
\end{bmatrix}.
\]

Therefore, 
\[
E_0(z^3) = \frac{1}{3} [H_1(z) + H_1(zW_3^1) + H_1(zW_3^2)]
\]
\[
\begin{align*}
E_1(z^3) &= z^{-1}E_1(z^3) = \frac{1}{3} [H_2(z) + W_1^2H_2(zW_3^1) + W_1H_2(zW_3^2)] = \frac{1}{3} \left( \frac{4 + 2.1e^{j2\pi/3}z^{-1} - 3.4e^{j4\pi/3}z^{-2}}{1 - 0.8e^{j2\pi/3}z^{-1} + 0.6e^{j4\pi/3}z^{-2}} + \frac{4 + 2.1e^{j4\pi/3}z^{-1} - 3.4e^{j2\pi/3}z^{-2}}{1 - 0.8e^{j4\pi/3}z^{-1} + 0.6e^{j2\pi/3}z^{-2}} \right) \\
&= \frac{4 - 0.716z^{-3} - 1.224z^{-6}}{1 + 0.928z^{-3} + 0.216z^{-6}}, \\
E_2(z^3) &= z^{-1}E_2(z^3) = \frac{1}{3} [H_2(z) + W_1^2H_2(zW_3^1) + W_1H_2(zW_3^2)] = \frac{1}{3} \left( \frac{4 + 2.1e^{j2\pi/3}z^{-1} - 3.4e^{j4\pi/3}z^{-2}}{1 - 0.8e^{j2\pi/3}z^{-1} + 0.6e^{j4\pi/3}z^{-2}} + \frac{4 + 2.1e^{j4\pi/3}z^{-1} - 3.4e^{j2\pi/3}z^{-2}}{1 - 0.8e^{j4\pi/3}z^{-1} + 0.6e^{j2\pi/3}z^{-2}} \right) \\
&= \frac{-1.56z^{-3} - 0.876z^{-6}}{1 + 0.928z^{-3} + 0.216z^{-6}}, \\
Hence, \ E_0(z) &= \frac{4 - 0.716z^{-1} - 1.224z^{-2}}{1 + 0.928z^{-1} + 0.216z^{-2}}, \ E_1(z) = \frac{-1.56z^{-1} - 0.876z^{-2}}{1 + 0.928z^{-1} + 0.216z^{-2}}, \\
E_2(z) &= \frac{5.3z^{-1} + 2.312z^{-2}}{1 + 0.928z^{-1} + 0.216z^{-2}}.
\end{align*}
\]

**14.4** \( H_0(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5} \), and \( H_1(z) = H_0(-z) = h[0] - h[1]z^{-1} + h[2]z^{-2} - h[3]z^{-3} + h[4]z^{-4} - h[5]z^{-5} \). A realization of \( H_0(z) \) and \( H_1(z) \) in the form of Figure P14.1 using 5 delays and 6 multipliers is shown below:

![Diagram of a signal flow graph](image)

$H_1(z) = H_0(-z) = h[0] - h[1]z^{-1} + h[2]z^{-2} - h[3]z^{-3} + h[4]z^{-4} - h[5]z^{-5}$. A realization of $H_0(z)$ and $H_1(z)$ in the form of Figure P14.1 using 6 multipliers is shown below:

14.6 (a) The structure of Figure P14.2(a) with internal variables labeled is shown below:

Analyzing the above structure we arrive at

$$\begin{bmatrix} V_0(z) \\ V_1(z) \\ V_2(z) \\ V_3(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} X_0(z) \\ X_1(z) \\ X_2(z) \\ X_3(z) \end{bmatrix},$$

and $Y(z) = R_3(z^4)V_3(z) + z^{-1}R_2(z^4)V_2(z) + z^{-2}R_1(z^4)V_1(z) + z^{-3}R_0(z^4)V_0(z)$.

The first equation leads to

$V_0(z) = X_0(z) + X_1(z) + X_2(z) + X_3(z)$,  
$V_1(z) = X_0(z) - jX_1(z) - X_2(z) + jX_3(z)$,  
$V_2(z) = X_0(z) - X_1(z) + X_2(z) - X_3(z)$,  
$V_4(z) = X_0(z) + jX_1(z) - X_2(z) - jX_3(z)$.

Substituting these 4 equations in the equation for $Y(z)$ and solving for the transfer functions $G_i(z) = Y(z)/X_i(z)$, $0 \leq i \leq 3$, we arrive at

$G_0(z) = R_3(z^4) + z^{-1}R_2(z^4) + z^{-2}R_1(z^4) + z^{-3}R_0(z^4)$,  
$G_1(z) = jR_3(z^4) - z^{-1}R_2(z^4) - jz^{-2}R_1(z^4) + z^{-3}R_0(z^4)$,  
$G_2(z) = -R_3(z^4) + z^{-1}R_2(z^4) - z^{-2}R_1(z^4) + z^{-3}R_0(z^4)$,  
$G_3(z) = -jR_3(z^4) - z^{-1}R_2(z^4) + jz^{-2}R_1(z^4) + z^{-3}R_0(z^4)$. 

Not for sale
Substituting the expressions for $R_i(z)$, $0 \leq i \leq 3$, we finally arrive at

\[
G_0(z) = 2 + z^{-1} + 4z^{-2} + z^{-3} - 1.5z^{-4} + 0.3z^{-5} - 0.9z^{-6} + 3.7z^{-7} + 3.1z^{-8} - 0.8z^{-9} + 2.3z^{-10} + 1.7z^{-11},
\]

\[
G_1(z) = j2 - z^{-1} - j4z^{-2} + 1.5z^{-3} - j1.5z^{-4} - 0.3z^{-5} + j0.9z^{-6} + 3.7z^{-7} + j3.1z^{-8} - 0.8z^{-9} - j2.3z^{-10} + 1.7z^{-11},
\]

\[
G_2(z) = -2 + z^{-1} - 4z^{-2} + z^{-3} + 1.5z^{-4} + 0.3z^{-5} + 0.9z^{-6} + 3.7z^{-7} - 3.1z^{-8} - 0.8z^{-9} - 2.3z^{-10} + 1.7z^{-11},
\]

\[
G_3(z) = -j2 - z^{-1} + j4z^{-2} + z^{-3} + j1.5z^{-4} - 0.3z^{-5} - j0.9z^{-6} + 3.7z^{-7} - j3.1z^{-8} + 0.8z^{-9} + j2.3z^{-10} + 1.7z^{-11}.
\]

**b)** The magnitude responses of all 4 analysis filters are shown below:

**14.7** $Y(z) = [H_0(z)G_0(z) + H_1(z)G_1(z)]X(z)$. Now, $H_0(z) = \frac{1 + z^{-1}}{2}$ and $H_1(z) = \frac{1 - z^{-1}}{2}$.

Choose $G_0(z) = \frac{1 + z^{-1}}{2}$ and $G_1(z) = \frac{1 - z^{-1}}{2}$. Then,

\[
Y(z) = \left(\frac{1}{4}(1 + z^{-1})^2 - \frac{1}{4}(1 - z^{-1})^2\right)X(z) = \frac{1}{4}(1 + 2z^{-1} + z^{-2} - 1 + 2z^{-1} - z^{-2})X(z) = z^{-1}X(z).
\]

Or in other words, $y[n] = x[n - 1]$ indicating that the structure of Figure P14.3 is a perfect reconstruction system with the above choices of filters.

**14.8** (a) Since $H_0(z)$ and $H_1(z)$ are power-complementary,

\[
H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = 1.
\]

Now,

\[
Y(z) = (H_0(z)G_0(z) + H_1(z)G_1(z))X(z)
\]

\[
= z^{-N}[H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1})]X(z) = z^{-N}X(z).
\]

Or in other words, $y[n] = x[n - N]$ indicating that the structure of Figure P14.3 is a perfect reconstruction system with the above choices of filters.
(b) If $H_0(z)$ and $H_1(z)$ are causal FIR transfer functions of order $N$ each, $H_0(z)$ and $H_1(z)$ are polynomials in $z^{-1}$. As a result, $H_0(z^{-1})$ and $H_1(z^{-1})$ are polynomials in $z$ with the highest power being $z^N$. Hence, $z^{-N}H_0(z^{-1})$ and $z^{-N}H_1(z^{-1})$ are polynomials in $z^{-1}$, making the synthesis filters $G_0(z)$ and $G_1(z)$ causal FIR transfer functions of order $N$ each.

(c) From Figure P14.3, for perfect reconstruction we require $H_0(z)G_0(z) + H_1(z)G_1(z) = z^{-N}$. From Part (a) we note that the perfect reconstruction condition is satisfied with $G_0(z) = z^{-N}H_0(z^{-1})$ and $G_1(z) = z^{-N}H_1(z^{-1})$, if $H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = 1$, i.e., if $H_0(z)$ and $H_1(z)$ are power-complementary. The last condition is satisfied if and only if

$$H_0(z) = \frac{z^{-N/2}}{2}(z^{-n_o} + z^{n_o})$$

and

$$H_1(z) = \frac{z^{-N/2}}{2}(z^{-n_o} - z^{n_o}).$$

As a result, $G_0(z)$ and $G_1(z)$ are also of the form

$$G_0(z) = \frac{z^{-N/2}}{2}(z^{-n_o} + z^{n_o})$$

and

$$G_1(z) = \frac{z^{-N/2}}{2}(z^{-n_o} - z^{n_o}).$$

14.9 (a) $G(z) = \frac{0.0985(1 + z^{-1})^3}{(1 - 0.1584z^{-1})(1 - 0.4189z^{-1} + 0.3554z^{-2})}$

$$= \frac{1}{2} \left[ \frac{0.3554 - 0.4189z^{-1} + z^{-2}}{1 - 0.4189z^{-1} + 0.3554z^{-2}} + \frac{-0.1584 + z^{-1}}{1 - 0.1584z^{-1}} \right].$$

Hence,

$$\mathcal{A}_0(z) = \frac{0.3554 - 0.4189z^{-1} + z^{-2}}{1 - 0.4189z^{-1} + 0.3554z^{-2}}$$

and

$$\mathcal{A}_1(z) = \frac{-0.1584 + z^{-1}}{1 - 0.1584z^{-1}}.$$

(b) $G(z)$ can be realized with only 3 multipliers as a parallel connection of two allpass filters in the form of Figure 8.43, where the allpass filter $\mathcal{A}_0(z)$ is realized using any one of the Type 2 or Type 3 second-order allpass structures requiring 2 multipliers and the allpass filter $\mathcal{A}_1(z)$ is realized using any one of the Type 1 first-order allpass structures requiring 1 multiplier.

(c) $H(z) = \frac{1}{2}(\mathcal{A}_0(z) - \mathcal{A}_1(z)) = \frac{0.2564(1 - z^{-1})^3}{(1 - 0.1584z^{-1})(1 - 0.4189z^{-1} + 0.3554z^{-2})}.$

(d) A plot of the magnitude responses of $G(z)$ and $H(z)$ is shown on top of the next page.
14.10 (a) \[ G(z) = \frac{0.1868(1 + 1.0902z^{-1} + 1.0902z^{-2} + z^{-3})}{(1 - 0.3628z^{-1})(1 - 0.5111z^{-1} + 0.7363z^{-2})} \]
\[ = \frac{1}{2} \left[ \frac{0.7363 - 0.5111z^{-1} + z^{-2}}{1 - 0.5111z^{-1} + 0.7363z^{-2}} + \frac{-0.3628 + z^{-1}}{1 - 0.3628z^{-1}} \right]. \]
Hence,
\[ \mathcal{A}_0(z) = \frac{0.7363 - 0.5111z^{-1} + z^{-2}}{1 - 0.5111z^{-1} + 0.7363z^{-2}} \]
and
\[ \mathcal{A}_1(z) = \frac{-0.3628 + z^{-1}}{1 - 0.3628z^{-1}}. \]

(b) \( G(z) \) can be realized with only 3 multipliers as a parallel connection of two allpass filters in the form of Figure 8.43, where the allpass filter \( \mathcal{A}_0(z) \) is realized using any one of the Type 2 or Type 3 second-order allpass structures requiring 2 multipliers and the allpass filter \( \mathcal{A}_1(z) \) is realized using any one of the Type 1 first-order allpass structures requiring 1 multiplier.

c) \[ H(z) = \frac{1}{2} \left[ \mathcal{A}_0(z) - \mathcal{A}_1(z) \right] = \frac{0.5495(1 - 1.7866z^{-1} - 1.7866z^{-2} + z^{-3})}{(1 - 0.3628z^{-1})(1 - 0.5111z^{-1} + 0.7363z^{-2})}. \]

(d) A plot of the magnitude responses of \( G(z) \) and \( H(z) \) is shown below:

14.11 An \( N \)-th order, with \( N \) odd, elliptic lowpass transfer function \( H(z) \) satisfies the condition
\[ H(z)H(z^{-1}) = \frac{1}{1 + \varepsilon^2 R_N(z)R_N(z^{-1})}, \] (A)

where \( R_N(z) \) is a rational function of the form

\[ R_N(z) = \left( \frac{1-z^{-1}}{1-z^{-1}} \right)^{N-1} \prod_{\ell=0}^{2} \left( \frac{1-z^{-1}e^{ji\phi_\ell}}{1-z^{-1}e^{ji\zeta_\ell}} \right). \] (B)

In the above equation, the frequencies \( \zeta_\ell \) are the transmission zeros at which \( H(e^{j\omega}) \) is equal to 0, i.e., \( H(e^{j\xi_\ell}) = 0 \), and the frequencies \( \phi_\ell \) are the reflection zeros at which \( |H(e^{j\omega})| \) is equal to the maximum value of 1, i.e., \( |H(e^{j\phi_\ell})| = 1 \).

From Eq. (B) it follows that

\[ R_N(z) = -R_N(z^{-1}). \] (C)

Now, as \( H(z) \) satisfies the power-symmetric condition, we have

\[ H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1, \] i.e.,

\[ H(z)H(z^{-1}) = 1 - H(-z)H(-z^{-1}). \] (D)

From the above equation it follows that the transmission zeros of \( H(-z) \) are at frequencies \( \pi - \phi_\ell \) and its reflection zeros are at \( \pi - \xi_\ell \). As a result, \( \phi_\ell + \xi_\ell = \pi \).

Hence, it follows that \( R_N(-z) = 1/R_N(z) \), or equivalently,

\[ R_N(z)R_N(-z) = 1. \] (E)

From Eq. (A) we have \( H(-z)H(-z^{-1}) = \frac{1}{1 + \varepsilon^2 R_N(-z)R_N(-z^{-1})} \), which when substituted in Eq. (D) yields

\[ H(z)H(z^{-1}) = 1 - \frac{1}{1 + \varepsilon^2 R_N(-z)R_N(-z^{-1})} = \frac{1}{1 + \frac{1}{\varepsilon^2 R_N(-z)R_N(-z^{-1})}}. \] In view of Eq. (E), the above equation reduces to

\[ H(z)H(z^{-1}) = \frac{1}{1 + \frac{1}{\varepsilon^2 R_N(z)R_N(z^{-1})}}. \] (F)

Comparing Eqs. (A) and (F) we thus conclude that \( \varepsilon^2 = 1 \), and hence,

\[ H(z)H(z^{-1}) = \frac{1}{1 + R_N(z)R_N(z^{-1})}. \] Now at a pole \( z = \lambda \) of \( H(z) \),

\[ R_N(\lambda)R_N(\lambda^{-1}) = -1. \] From this relation, and Eqs. (C) and (E), it follows then that

\[ \left| \frac{R_N(\lambda)}{R_N(-\lambda)} \right| = 1. \] Also, for a lowpass power-symmetric transfer function, \( \xi_\ell > \pi / 2 \)
and $\phi < \pi/2$. Consequently, the poles of the rational function $R_N(z)/R_N(-z)$ must lie on the left half of the $z$-plane.

Since, the magnitude of the rational function $R_N(z)/R_N(-z)$ on the imaginary axis is 1, using maximum-modulus theorem it can be shown that $|R_N(z)/R_N(-z)| < 1$ for $\text{Re } z > 0$. In a similar manner by replacing $z$ with $-z$, it can be shown that $|R_N(-z)/R_N(z)| < 1$ for $\text{Re } z > 0$, or, equivalently, $|R_N(z)/R_N(-z)| > 1$ for $\text{Re } z < 0$. Thus,

\[
\begin{align*}
\frac{R_N(z)}{R_N(-z)} &< 1, \quad \text{Re } z > 0, \\
1 &\quad \text{Re } z = 0, \\
\frac{R_N(z)}{R_N(-z)} &> 1, \quad \text{Re } z < 0,
\end{align*}
\]

or, in other words, all poles of $H(z)$ lie on the imaginary axis of the $z$-plane.

14.12 Now the magnitude-square function of an $N$-th order analog lowpass Butterworth function $G_a(s)$ is given by $|G_a(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$, where $\Omega_c$ is the 3-dB cutoff angular frequency. For $\Omega_c = 1$, then $|G_a(j\Omega)|^2 = \frac{1}{1 + \Omega^{2N}}$. The corresponding transfer function of an $N$-th order analog highpass Butterworth function is simply $G_a(1/s)$, whose the magnitude-square function is given by

\[
|G_a\left(\frac{1}{j\Omega}\right)|^2 = \frac{\Omega^{2N}}{1 + \Omega^{2N}}.
\]

As a result,

\[
|G_a(j\Omega)|^2 + |G_a\left(\frac{1}{j\Omega}\right)|^2 = \frac{1}{1 + \Omega^{2N}} + \frac{\Omega^{2N}}{1 + \Omega^{2N}} = 1.
\]

Now the bilinear transformation maps the analog angular frequency $\Omega$ to the digital angular frequency $\omega$ through the relation $e^{j\omega} = \frac{1 - j\Omega}{1 + j\Omega}$. As $-e^{j\omega} = \frac{1 - (1/j\Omega)}{1 + (1/j\Omega)}$, the analog angular frequency $1/\Omega$ is mapped to the digital angular frequency $\pi + \omega$. Hence, the relation

\[
|G_a(j\Omega)|^2 + |G_a\left(\frac{1}{j\Omega}\right)|^2 = 1, \text{ becomes } \left|H_0(e^{j\omega})\right|^2 + \left|H_0(e^{j(\pi + \omega)})\right|^2 = \left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 1,
\]

where $H_0(e^{j\omega})$ is the frequency response of the digital lowpass filter $H_0(z)$ obtained by applying the bilinear transformation to $G_a(s)$ and $H_1(e^{j\omega})$ is the frequency response of the digital highpass filter $H_0(z)$ obtained by applying the bilinear transformation to $G_a(1/s)$. Note that a digital transfer function satisfying the condition $\left|H_0(e^{j\omega})\right|^2 + \left|H_1(e^{j\omega})\right|^2 = 1$, is called power-symmetric.
Moreover, from the relation it follows that the analog 3-dB cutoff angular frequency \( \Omega_c = 1 \) is mapped into the digital 3-dB cutoff angular frequency \( \omega_c = \pi / 2. \) Hence, \( H_0(z) \) is a digital half-band lowpass filter.

Let \( H_0(z) = \frac{P_0(z)}{D_0(z)} \), where \( P_0(z) \) and \( D_0(z) \) are polynomials in \( z^{-1} \). Hence,

\[
H_0(e^{j\omega}) = \frac{P_0(e^{j\omega})}{D_0(e^{j\omega})} \quad \text{and} \quad H_0(-e^{j\omega}) = \frac{P_0(-e^{j\omega})}{D_0(-e^{j\omega})}.
\]

Now,

\[
\left| H_0(-e^{j\omega}) \right|^2 = \frac{P_0(-e^{j\omega})P_0^*(e^{j\omega})}{D_0(-e^{j\omega})D_0^*(-e^{j\omega})} = 1 - \left| H_0(e^{j\omega}) \right|^2 = 1 - \frac{P_0(e^{j\omega})P_0^*(e^{j\omega})}{D_0(e^{j\omega})D_0^*(e^{j\omega})}
\]

\[
= \frac{D_0(e^{j\omega})D_0^*(e^{j\omega}) - P_0(e^{j\omega})P_0^*(e^{j\omega})}{D_0(e^{j\omega})D_0^*(e^{j\omega})}.
\]

Note that there are no common factors between \( P_0(e^{j\omega}) \) and \( D_0(e^{j\omega}) \), and between \( P_0^*(e^{j\omega}) \) and \( D_0^*(e^{j\omega}) \). As a result, there are no common factors between \( P_0(-e^{j\omega}) \) and \( D_0(-e^{j\omega}) \). This means then \( D_0(e^{j\omega})D_0^*(e^{j\omega}) = D_0(-e^{j\omega})D_0^*(-e^{j\omega}) \). Consequently, \( D_0(e^{j\omega}) = D_0(-e^{j\omega}) \), or \( D_0(e^{j\omega}) = d_0(e^{j2\omega}) \). Hence, \( D_0(z) = d_0(z^2) \). Since \( H_0(z) = \frac{P_0(z)}{D_0(z)} = \frac{P_0(z)}{d_0(z^2)} \), it follows then \( H_1(z) = \frac{P_0(-z)}{d_0(z^2)} \). We have shown earlier that \( H_0(z) \) and \( H_1(z) \) are power-complementary. Also, \( P_0(z) \) is a symmetric polynomial of odd order and \( P_1(z) \) is an anti-symmetric polynomial of odd order. As a result, we can express

\[
H_0(z) = \frac{1}{2} [A_0(z) + A_1(z)] \quad \text{and} \quad H_1(z) = \frac{1}{2} [A_0(z) - A_1(z)] ,
\]

where \( A_0(z) \) and \( A_1(z) \) are stable allpass functions for stable \( H_0(z) \) and \( H_1(z) \). But, \( H_1(z) = H_0(-z) \). Hence, \( H_0(z) = \frac{1}{2} [A_0(-z) - A_1(-z)] \). It therefore follows that \( A_0(z) = A_0(-z) = A_0(z^2) \), and \( A_1(z) = -A_1(-z) = z^{-1}A_1(z^2) \). Thus, \( H_0(z) = \frac{1}{2} [A_0(z^2) + z^{-1}A_1(z^2)] \).

14.13 From Eq. (14.18), \( Y(z) = T(z)X(z) + A(z)X(-z) \). Let \( Z^{-1}\{T(z)\} = t[n] \), and \( Z^{-1}\{A(z)\} = a[n] \). Then, an inverse z-transform of Eq. (14.18) yields

\[
y[n] = \sum_{\ell=-\infty}^{\infty} t[\ell]x[n-\ell] + \sum_{\ell=-\infty}^{\infty} a[\ell](-1)^{n-\ell} x[n-\ell] = \sum_{\ell=-\infty}^{\infty} (t[\ell] + (-1)^{n-\ell} a[\ell]) x[n-\ell].
\]

Define \( f_0[n] = t[n] + (-1)^n a[n] \) and \( f_1[n] = t[n] - (-1)^n a[n] \). Then we can write
\[
\begin{align*}
    y[n] &= \begin{cases} 
    \sum_{\ell=-\infty}^{\infty} f_0[\ell] x[n-\ell], & \text{for } n \text{ even}, \\
    \sum_{\ell=-\infty}^{\infty} f_1[\ell] x[n-\ell], & \text{for } n \text{ odd.}
\end{cases}
\end{align*}
\]

The corresponding equivalent realization of the 2-channel QMF bank is therefore as indicated below:

As can be seen from the above representation, the QMF bank is, in general, a time-varying system with a period 2. Note that \( A(z) = 0 \), if then it becomes a linear, time-invariant system.

**14.14** \( G_a(s) = \frac{1}{(s+1)(s^2+s+1)} \). The digital transfer function \( H_0(z) \) obtained by a bilinear transformation is given by

\[
H_0(z) = G_a(s)|_{s=\frac{z^{-1}-1}{z+1}} = \frac{(z+1)^3}{(z-1+z-1)(z-1)^2 + (z^2-1) + (z+1)^2} = \frac{(z+1)^3}{2z(3z^2+1)}
\]

\[
= \frac{(1+z^{-1})^3}{6+2z^{-2}} = \frac{1+3z^{-1}+3z^{-2}+z^{-3}}{6+2z^{-2}} = \frac{1}{2} \left( \frac{1+3z^{-2}+z^{-1}}{3+z^{-2}+z^{-1}} \right)
\]

\[= \frac{1}{2} \left[ A_0(z^2) + z^{-1} A_1(z^2) \right],
\]

where \( A_0(z) = \frac{1+3z^{-1}}{3+z^{-1}} \) and \( A_1(z) = 1 \).

The corresponding power-complementary transfer function is given by

\[
H_1(z) = \frac{1}{2} \left[ A_0(z^2) - z^{-1} A_1(z^2) \right] = \frac{1}{2} \left( \frac{1+3z^{-2}+z^{-1}}{3+z^{-2}} - z^{-1} \right) = \frac{(1-z^{-1})^3}{6+2z^{-2}}.
\]

A realization of the analysis part of the QMF bank is as shown in Figure 14.11 where the first-order allpass transfer function \( A_0(z) \) is realized using any one of the single-multiplier structure of Figure 8.24 and the zero-th order allpass transfer function \( A_1(z) \) is replaced with a direct connection between the input and the output.

**14.15** \( G_a(s) = \frac{1}{s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1} \). The digital transfer function \( H_0(z) \) obtained by a bilinear transformation is given by

\[
H_0(z) = \frac{(1+z^{-1})^5}{18.9443 + 12z^{-2} + 1.0557z^{-4}} = \frac{0.0528(1+z^{-1})^5}{1 + 0.6334z^{-2} + 0.0557z^{-4}}
\]
\[ H(z) = \frac{0.0528 + 0.5279 z^{-2} + 0.2639 z^{-4}}{1 + 0.6334 z^{-2} + 0.0557 z^{-4}} + \frac{+0.2639 z^{-1} + +0.5279 z^{-3} + 0.0528 z^{-5}}{1 + 0.6334 z^{-2} + 0.0557 z^{-4}} \]
\[ = \frac{0.0528(1 + 9.4704 z^{-2})}{1 + 0.1056 z^{-2}} + z^{-1} \frac{0.2639(1 + 1.8948 z^{-2})}{1 + 0.5278 z^{-2}} \]
\[ = \frac{1}{2} \left[ A_0(z^2) + z^{-1} A_1(z^2) \right], \text{where } A_0(z) = \frac{0.1056 + z^{-1}}{1 + 0.1056 z^{-1}} \text{ and } A_1(z) = \frac{0.5278 + z^{-1}}{1 + 0.5278 z^{-1}}. \]

Its power-complementary transfer function is given by
\[ H_1(z) = \frac{1}{2} \left[ A_0(z^2) - z^{-1} A_1(z^2) \right] = \frac{1}{2} \left[ \frac{0.1056 + z^{-2}}{1 + 0.1056 z^{-2}} - z^{-1} \frac{0.5278 + z^{-2}}{1 + 0.5278 z^{-2}} \right]. \]

In the realization of a magnitude-preserving QMF bank as shown in Figure 14.11, the realization of the allpass filters \( A_0(z) \) and \( A_1(z) \) require 1 multiplier each, and hence, the realization of the analysis (and the synthesis) filter bank requires a total of 2 multipliers.

14.16 (a) Total number of multipliers required is \( 4(2N - 1) \). Hence, the total number of multiplications per second is equal to \( 4(2N - 1)F_T = 4(2N - 1)/T \), where \( F_T = 1/T \) is the sampling frequency in Hz.

(b) In Figure 14.11, \( H_0(z) = \frac{1}{2} \left[ A_0(z^2) + z^{-1} A_1(z^2) \right] \) and \( H_1(z) = \frac{1}{2} \left[ A_0(z^2) - z^{-1} A_1(z^2) \right]. \) If the order of \( A_0(z) \) is \( K \) and the order of \( A_1(z) \) is \( L \), then the order of \( H_0(z) \) is \( 2K + 2L + 1 = N \). Hence, \( K + L = (N - 1)/2 \). The total number of multipliers needed to implement is \( K \), while the total number of multipliers needed to implement is \( L \). Hence, the total number of multipliers required to implement the QMF bank of Figure 14.11 is \( 2(K + L) = N - 1 \). However, the multipliers here are operating at half of the sampling rate of the input \( x[n] \). As a result, the total number of multiplications per second in this case is \( (N - 1)F_T / 2 = (N - 1)/2T \).

14.17 The 4 filters of the 2-channel QMF bank are: \( H_0(z) = 4z^{-2} \), \( H_1(z) = z^{-1} \), \( G_0(z) = 0.25z^{-1} \), \( G_1(z) = z^{-1} \). Substituting these transfer functions in Eq. (14.21) we get
\[ H_0(-z)G_0(z) + H_1(-z)G_1(z) = 4z^{-2} \times 0.25z^{-1} - z^{-1} \times z^{-2} = z^{-3} - z^{-3} = 0, \text{ implying aliasing cancellation condition holds.} \] Next, substituting the transfer functions in Eq. (14.27) we get
\[ H_0(z)G_0(z) + H_1(z)G_1(z) = 4z^{-2} \times 0.25z^{-1} + z^{-1} \times z^{-2} = 2z^{-3}. \] Hence, it is a perfect reconstruction QMF bank.

14.18 The aliasing cancellation condition is satisfied if the synthesis filters satisfy Eq. (14.37). Hence, we choose \( H_1(z) = G_0(-z) = d - e z^{-1} + f z^{-2} - g z^{-3} + h z^{-4}, \) and \( G_1(z) = -H_0(-z) = -(a - b z^{-1} + c z^{-2}) = -a - b z^{-1} - c z^{-2}. \)

14.19 The 4 filters satisfy the aliasing cancellation condition of Eq. (14.21) and the perfect reconstruction condition of Eq. (14.27). Interchanging \( H_0(z) \) and \( G_0(z), \) and interchanging \( H_1(z) \) and \( G_1(z) \) in Eq. (14.21) we arrive at the new aliasing term \( A(z) = G_0(-z)H_0(z) + G_1(-z)H_1(z). \) Hence, \( A(-z) = G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0, \) implying that the aliasing condition is still satisfied after interchanging the analysis and synthesis filters. It also follows from Eq. (14.27), an interchange of the analysis and synthesis filters does not change the expression for the distortion transfer function and as a result the perfect reconstruction condition is still satisfied.

14.20 From Eq. (14.73), it can be seen that the normalized product filter \( P(z) \) is of odd length \( 2L + 1. \) Let the degrees of \( H_0(z) \) and \( G_0(z) \) be \( N \) and \( K, \) respectively. Hence, it follows from Eq. (14.76), \( N + K = 2L. \) Therefore, either \( N \) and \( K \) are both even or both odd. As a result, \( H_0(z) \) and \( G_0(z) \) cannot be of even and odd lengths, respectively.

Moreover, \( P(z) \) is a symmetric polynomial. If \( H_0(z) \) is a symmetric polynomial and \( G_0(z) \) is an antisymmetric polynomial, their product cannot be a symmetric polynomial.

14.21 \( P(z) = (1 + z^{-1})^3(1 + z)^3(az^2 + bz + c + bz^{-1} + az^{-2}) = az^5 + (6a + b)z^4 + (15a + 6b + c)z^3 + (20a + 16b + 6c)z^2 + (16a + 26b + 15c)z + (12a + 30b + 20c) + (16a + 26b + 15c)z^{-1} + (20a + 16b + 6c)z^{-2} + (15a + 6b + c)z^{-3} + (6a + b)z^{-4} + az^{-5}. \)

Since the even powers of \( P(z) \) must be zeros and the coefficient of \( z^0 \) be equal to 1, we must have \( 6a + b = 0, 20a + 16b + 6c = 0, \) and \( 12a + 30b + 20c = 1. \) Solving these 3 equations we arrive at \( a = 0.01171875, b = -0.0703125, c = 0.1484375. \) Hence, \( P(z) = (1 + z^{-1})^3(1 + z)^3(0.1171875z^2 - 0.0703125z + 0.1484375 - 0.0703125z^{-1} + 0.1171875z^{-2}) = 0.1171875z^5(1 + z^{-1})^6(1 - 5.4255z^{-1} + 9.4438z^{-2}) \times (1 - 0.5745z^{-1} + 0.1059z^{-2}), \)

One possible factorization of \( P(z) \) is given by

Not for sale
\[ H_0(z) = 0.01171875(1 + z^{-1})(1 - 5.4255z^{-1} + 9.4438z^{-2})(1 - 0.5745z^{-1} + 0.1059z^{-2}), \]
and \[ G_0(z) = (1 + z^{-1})^5. \] Thus, the highpass analysis filter is given by \[ H_1(z) = G_0(-z) = (1 - z^{-1})^5. \]

14.22 Without any loss of generality, let \( N = 4 \). Let
Then,
\[ H_1(z) = z^{-4}H_0(-z^{-1}) = z^{-4}(h[0] - h[1]z + h[2]z^2 - h[3]z^3 + h[4]z^4) \]
\[ = h[4] - h[3]z^{-1} + h[2]z^{-2} - h[1]z^{-3} + h[0]z^{-4}. \] A realization of the analysis filter bank using \( N + 1 = 5 \) multipliers and \( 2N = 8 \) two-input adders is shown on top of the next page:

14.23 For an orthogonal filter bank, the two analysis filters \( H_0(z) = \sum_{n=0}^{N} h_0[n]z^{-n} \) and \( H_1(z) = \sum_{n=0}^{N} h_1[n]z^{-n} \) are causal FIR filters satisfy the power complementary property, i.e., \( H_0(z)H_0(z^{-1}) + H_1(z)H_1(z^{-1}) = \gamma, \) where \( \gamma > 0 \).
We assume \( h_0[0] \neq 0 \) and \( h_1[0] \neq 0 \). It follows from Section 7.3 that a linear-phase FIR filter has either a symmetric or an anti-symmetric impulse response. As a result, if \( H_0(z) \) and \( H_1(z) \) are linear-phase transfer functions, then they are of the form \( H_0(z) = e^{-j\beta_0} z^{-N}H_0(z^{-1}) \) and \( H_1(z) = e^{-j\beta_0} z^{-N}H_1(z^{-1}). \)
Substituting the expressions for \( H_0(z^{-1}) \) and \( H_1(z^{-1}) \) in the power-complementary condition we arrive at \( e^{j\beta_0 z^{-N}}H_0^2(z) + e^{j\beta_1 z^{-N}}H_1^2(z) = \gamma \), which can be rewritten as
\[ \left[ e^{j\beta_0 / 2}H_0(z) + je^{j\beta_1 / 2}H_1(z) \right] \left[ e^{j\beta_0 / 2}H_0(z) - je^{j\beta_1 / 2}H_1(z) \right] = \gamma z^{-N}. \] Both quantities inside the square brackets are causal FIR transfer functions. Hence, we have
\[ e^{j\beta_0 / 2}H_0(z) + je^{j\beta_1 / 2}H_1(z) = \alpha_0 z^{-N_0}, \]
and
\[ e^{j\beta_0 / 2}H_0(z) - je^{j\beta_1 / 2}H_1(z) = \alpha_1 z^{-N_1}, \]
where $\alpha_0\alpha_1 = \gamma$ and $N_0 + N_1 = N$. Adding Eq. (H) from Eq. (G) we get
\[ H_0(z) = e^{-j\beta_0/2} (\alpha_0 z^{-N_0} + \alpha_1 z^{-N_1}) \] and
\[ H_1(z) = -j e^{-j\beta_1/2} (\alpha_0 z^{-N_0} - \alpha_1 z^{-N_1}). \] Hence, as the analysis filters have transfer functions that are weighted sum of two delays, they can not be used to design filters with any practical frequency response specifications.

**14.24** From Eq. (14.90) we have
\[ H_0(z) = -z^{-3} H_1(-z^{-1}) = -z^{-3} (a - bz + cz^2 - dz^3) = d - cz^{-1} + bz^{-2} - az^{-3}. \] Next, from Eq. (14.92) we get
\[ G_0(z) = z^{-3} H_0(z^{-1}) = z^{-3} (d - cz + bz^2 - az^3) = -a + bz^{-1} - cz^{-2} + dz^{-3} \] and
\[ G_1(z) = z^{-3} H_1(z^{-1}) = z^{-3} (a + bz + cz^2 + dz^3) = d + cz^{-1} + bz^{-2} + az^{-3}. \]

**14.25** Substituting the transfer functions in Eq. (14.21) we get
\[ H_0(-z) G_0(z) + H_1(-z) G_1(z) = (3 - 4z^{-1})(-0.5 + z^{-1}) + (1 - 2z^{-1})(1.5 - 2z^{-1}) \]
\[ = (-1.5 + 5z^{-1} - 4z^{-2}) + (1.5 - 5z^{-1} + 4z^{-2}) = 0 \] implying that the aliasing cancellation condition is satisfied. Next, substituting the transfer functions in Eq. (14.27) we get
\[ H_0(z) G_0(z) + H_1(z) G_1(z) = (3 + 4z^{-1})(-0.5 + z^{-1}) + (1 + 2z^{-1})(1.5 - 2z^{-1}) \]
\[ = (-1.5 + z^{-1} + 4z^{-2}) + (1.5 + z^{-1} - 4z^{-2}) = 2z^{-1} \] implying that the perfect reconstruction condition is satisfied.

**14.26** $H_0(z) = a + bz^{-1} + cz^{-2} + dz^{-3} + ez^{-4} + fz^{-5}$. From Eq. (14.90) we get
\[ H_1(z) = z^{-5} H_0(-z^{-1}) = z^{-5} (a - bz + cz^2 - dz^3 + ez^4 - fz^5) \]
\[ = -f + ez^{-1} - dz^{-2} + cz^{-3} - bz^{-4} + az^{-5}. \] From Eq. (14.92), we arrive at
\[ G_0(z) = z^{-5} H_0(z^{-1}) = f + ez^{-1} + dz^{-2} + cz^{-3} + bz^{-4} + az^{-5} \] and
\[ G_1(z) = z^{-5} H_1(z^{-1}) = a - bz^{-1} + cz^{-2} - dz^{-3} + ez^{-4} - fz^{-5}. \]

**14.27** To develop the realization of the synthesis filter bank we redraw Figure P14.4 as shown below:

The $i$-th stage of the above analysis filter bank is of the form shown below:
The input-output relation of the lattice part of the above figure is given by
\[
\begin{bmatrix}
Y_0^{(i)}(z) \\
Y_1^{(i)}(z)
\end{bmatrix} = \begin{bmatrix}
1 & k_i \\
-k_i & 1
\end{bmatrix}
\begin{bmatrix}
V_0^{(i)}(z) \\
V_1^{(i)}(z)
\end{bmatrix}
\]
which can be solved for the input variables leading to
\[
\begin{bmatrix}
U_0^{(i)}(z) \\
U_1^{(i)}(z)
\end{bmatrix} = \begin{bmatrix}
1 & k_i \\
-k_i & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
Y_0^{(i)}(z) \\
Y_1^{(i)}(z)
\end{bmatrix} = \frac{1}{1+k_i^2}
\begin{bmatrix}
1 & -k_i \\
-k_i & 1
\end{bmatrix}
\begin{bmatrix}
Y_0^{(i)}(z) \\
Y_1^{(i)}(z)
\end{bmatrix},
\]
a realization of which is indicated below:

\[
y_0^{(i)}(z) \rightarrow \oplus \rightarrow y_0^{(i)}(z)
\]

\[
y_1^{(i)}(z) \rightarrow \oplus \rightarrow y_1^{(i)}(z)
\]

where \( s_i = k_i / (1+k_i^2) \). A cascade of the above two lattice structures thus has an input-output relation given by
\[
\begin{bmatrix}
U_0^{(i)}(z) \\
U_1^{(i)}(z)
\end{bmatrix} = \frac{1}{1+k_i^2}
\begin{bmatrix}
1 & k_i \\
-k_i & 1
\end{bmatrix}
\begin{bmatrix}
V_0^{(i)}(z) \\
V_1^{(i)}(z)
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
V_0^{(i)}(z) \\
V_1^{(i)}(z)
\end{bmatrix}.
\]

Likewise, the input-output relation of the delay part of the \( i \)-th stage of the analysis filter bank is given by
\[
\begin{bmatrix}
V_0^{(i)}(z) \\
V_1^{(i)}(z)
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & z^{-1}
\end{bmatrix}
\begin{bmatrix}
X_0^{(i)}(z) \\
X_1^{(i)}(z)
\end{bmatrix}
\]
which can be solved for the input variables leading to
\[
\begin{bmatrix}
W_0^{(i)}(z) \\
W_1^{(i)}(z)
\end{bmatrix} = \begin{bmatrix}
z^{-1} & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
V_0^{(i)}(z) \\
V_1^{(i)}(z)
\end{bmatrix},
\]
a realization of which is indicated below:

\[
v_0^{(i)}(z) \rightarrow \oplus \rightarrow w_0^{(i)}(z)
\]

\[
v_1^{(i)}(z) \rightarrow \oplus \rightarrow w_1^{(i)}(z)
\]

Hence, the cascaded-lattice realization of the synthesis filter bank of the two-channel orthogonal filter bank is as shown below where \( s_i = k_i / (1+k_i^2) \).:
14.28 Figure P14.4 with the down-samplers removed and internal variables labeled is shown below:

Denoting the transfer functions $H_0^{(r)}(z) = Y_0^{(r)}(z)/X(z)$ and $H_1^{(r)}(z) = Y_1^{(r)}(z)/X(z)$, we arrive at an equivalent representation of the above analysis filter bank as indicated below:

Analysis yields $H_0^{(N)}(z) = H_0^{(N-2)}(z) + k_N z^{-2} H_1^{(N-2)}(z)$ and $H_1^{(N)}(z) = -k_N H_0^{(N-2)}(z) + z^{-2} H_1^{(N-2)}(z)$. Solving these two equations, we get 

$$(1 + k_N^2) H_0^{(N-2)}(z) = H_0^{(N)}(z) - k_N H_1^{(N)}(z)$$ and 

$$(1 + k_N^2) z^{-2} H_1^{(N-2)}(z) = k_N H_0^{(N)}(z) + H_1^{(N)}(z).$$

Choose $k_N$ such that the highest power of $z^{-1}$ in $H_0^{(N)}(z) - k_N H_1^{(N)}(z)$ is eliminated. By construction, the next highest power of $z^{-1}$ also is removed reducing the order of $H_0^{(N-2)}(z)$ to $N - 2$. Moreover, the coefficients of $z^0$ and $z^{-1}$ in $k_N H_0^{(N)}(z) + H_1^{(N)}(z)$ are also zero for the above choice of $k_N$, resulting in a causal $H_1^{(N-2)}(z)$ of order $N - 2$. Continuing this process, we arrive at a cascaded lattice realization of the analysis filter bank of an orthogonal QMF bank.

From Eq. (14.87) we have $H_0^{(3)}(z) = 0.3415 + 0.5915z^{-1} + 0.1585z^{-2} - 0.0915z^{-3}$ and from Eq. (14.89a) we have $H_1^{(3)}(z) = -0.0915 - 0.1585z^{-1} + 0.5915z^{-2} - 0.3415z^{-3}$

We now form $(1 + k_3^2) H_0^{(1)}(z) = H_0^{(3)}(z) - k_3 H_1^{(3)}(z) = (0.3415 + 0.0915k_3) + (0.5915 + 0.1585k_3) z^{-1} + (0.1585 - 0.5915k_3) z^{-2} + (-0.0915 + 0.3415k_3) z^{-3}$.
Choose \( k_3 = 0.0915/0.3415 = 0.2679 \). Then \( 1.0718 H_0^{(1)}(z) = H_0^{(3)}(z) - k_3 H_1^{(3)}(z) = 0.366 + 0.634z^{-1} \), implying, \( H_0^{(1)}(z) = 0.3415 + 0.5915 z^{-1} = 0.3415(1+1.7321z^{-1}) \).

We next form \((1 + k_3^2)z^{-2} H_1^{(1)}(z) = \left[ k_3 H_0^{(3)}(z) + H_1^{(3)}(z) \right] \) or, 
\[
1.0718z^{-2} H_1^{(1)}(z) = \left[ 0.2679 H_0^{(3)}(z) + H_1^{(3)}(z) \right] = 0.634 - 0.366z^{-1} \), implying, 
\( H_1^{(1)}(z) = 0.5915 - 0.3415 z^{-1} = 0.3415(1.7321 - z^{-1}) \). Hence, \( k_1 = 1.7321 \). A cascaded lattice realization of the analysis filter bank is as shown below:

From the solution of Problem 14.27 we arrive at the structure of the synthesis filter bank shown on the next page to ensure the realization of the orthogonal perfect reconstruction QMF bank.

14.29 (a) \( H_0(z) = 1 + 3z^{-1} + 14z^{-2} + 22z^{-3} - 12z^{-4} + 4z^{-5} \). Hence, 
\( H_1(z) = -z^{-5} H_0(-z^{-1}) = 4 + 12z^{-1} + 22z^{-2} - 14z^{-3} + 3z^{-4} - z^{-5} \).

(b) To develop the cascade lattice realization of the analysis filter bank, we rewrite the analysis transfer functions as \( H_0^{(5)}(z) = H_0(z) \) and \( H_1^{(5)}(z) = H_1(z) \). Next, we determine \( H_0^{(3)}(z) \) using the relation \( (1 + k_5^2) H_0^{(3)}(z) = H_0^{(5)}(z) - k_5 H_1^{(5)}(z) \) and choose \( k_5 \) so that the coefficient of \( z^{-5} \) is eliminated. Now, 
\[
(1 + k_5^2) H_0^{(3)}(z) = (1 - 4k_5) + (3 - 12k_5)z^{-1} + (14 - 22k_5)z^{-2} + (22 + 14k_5)z^{-3} + (-12 - 3k_5)z^{-4} + (4 + k_5)z^{-5}.
\]
We choose \( k_5 = -4 \). Then \( (1 + k_5^2) H_0^{(3)}(z) = 17 H_0^{(3)}(z) = 17 + 51z^{-1} + 102z^{-2} + 34z^{-3} \). Note that as expected the above choice of \( k_5 \) also cancels the coefficient of \( z^{-4} \). We thus have,
\[
H_0^{(3)}(z) = 1 + 3z^{-1} + 6z^{-2} - 2z^{-3}.
\]
Next we determine \( H_1^{(3)}(z) \) using the relation \( (1 + k_5^2)z^{-2} H_1^{(3)}(z) = k_5 H_0^{(5)}(z) + H_1^{(5)}(z) \) which leads to 
\[
17z^{-2} H_1^{(3)}(z) = -34z^{-2} - 102z^{-3} + 51z^{-4} - 17z^{-5}.
\]
It should be noted that here,
as expected, the choice of $k_5$ has resulted in zero-valued coefficients of $z^0$ and $z^{-1}$. Hence, $H_1^{(3)}(z) = -2 - 6z^{-1} + 3z^{-2} - z^{-3}$. It can be verified that the same expression for $H_1^{(3)}(z)$ is obtained from $H_1^{(3)}(z) = -z^{-3}H_0^{(3)}(-z^{-1})$.

Next, we determine $H_0^{(1)}(z)$ using the relation 

$$(1 + k_5^2)H_0^{(1)}(z) = H_0^{(3)}(z) - k_3H_1^{(3)}(z)$$

and choose $k_3 = 2$ so that the coefficient of $z^{-3}$ is eliminated. This results in $H_0^{(1)}(z) = 1 + 3z^{-1}$ and $H_1^{(1)}(z) = 3 - z^{-1}$.

Analysis of the structure of Figure P14.4 yields $H_0^{(1)}(z) = 1 + k_1z^{-1}$ and $H_1^{(1)}(z) = k_1 - z^{-1}$. Hence, $k_1 = 3$. The final realization of the analysis filter bank is shown below:

From the solution of Problem 14.27 we arrive at the structure of the synthesis filter bank shown below to ensure the realization of a paraunitary perfect reconstruction QMF bank.

14.30 (a) $H_0(z) = 0.5 - z^{-1} + 10.5z^{-2} - 13.5z^{-3} - 5z^{-4} - 2.5z^{-5}$. Hence, $H_1(z) = -z^{-5}H_0(-z^{-1}) = -2.5 + 5z^{-1} - 13.5z^{-2} - 10.5z^{-3} - z^{-4} - 0.5z^{-5}$.

(b) To develop the cascade lattice realization of the analysis filter bank, we rewrite the analysis transfer functions as $H_0^{(5)}(z) = H_0(z)$ and $H_1^{(5)}(z) = H_1(z)$. Next, we determine $H_0^{(3)}(z)$ using the relation 

$$(1 + k_5^2)H_0^{(3)}(z) = H_0^{(5)}(z) - k_5H_1^{(5)}(z)$$

and choose $k_5$ so that the coefficient of $z^{-5}$ is eliminated. Now,

$$(1 + k_5^2)H_0^{(3)}(z) = (0.5 + 2.5k_5) + (-1 - 5k_5)z^{-1} + (10.5 + 13.5k_5)z^{-2} + (-13.5 + 10.5k_5)z^{-3} + (-5 + k_5)z^{-4} + (-2.5 + 0.5k_5)z^{-5}.$$ 

We choose $k_5 = 5$. Then

$$(1 + k_5^2)H_0^{(3)}(z) = 26H_0^{(3)}(z) = 13 - 26z^{-1} + 78z^{-2} + 39z^{-3}.$$ 

Note that as expected the above choice of $k_5$ also cancels the coefficient of $z^{-4}$. We thus have,
Next we determine \( H_1^{(3)}(z) \) using the relation \( (1 + k_5^2)z^{-2}H_1^{(3)}(z) = k_5H_0^{(3)}(z) + H_1^{(5)}(z) \) which leads to
\[
26z^{-2}H_1^{(3)}(z) = 39z^{-2} - 78z^{-3} - 26z^{-4} - 13z^{-5}.
\]
It should be noted that here, as expected, the choice of \( k_5 \) has resulted in zero-valued coefficients of \( z^0 \) and \( z^{-1} \). Hence, \( H_1^{(3)}(z) = 1.5 - 3z^{-1} - z^{-2} - 0.5z^{-3} \). It can be verified that the same expression for \( H_1^{(3)}(z) \) is obtained from \( H_1^{(3)}(z) = -z^{-3}H_0^{(3)}(-z^{-1}) \). Next, we determine \( H_0^{(1)}(z) \) using the relation \( (1 + k_3^2)H_0^{(1)}(z) = H_1^{(3)}(z) - k_3H_1^{(3)}(z) \) and choose \( k_3 = -3 \) so that the coefficient of \( z^{-3} \) is eliminated. This results in \( H_0^{(1)}(z) = 0.5 - z^{-1} = 0.5(1 - 2z^{-1}) \) and \( H_1^{(1)}(z) = -1 - 0.5z^{-1} = 0.5(-2 - z^{-1}) \).

Analysis of the structure of Figure P14.4 yields \( H_0^{(1)}(z) = 1 + k_1z^{-1} \) and \( H_1^{(1)}(z) = k_1 - z^{-1} \). Hence, \( k_1 = -2 \). The final realization of the analysis filter bank is shown on top of next page:

![Analysis Filter Bank](image)

From the solution of Problem 14.27 we arrive at the structure of the synthesis filter bank shown below to ensure the realization of a paraunitary perfect reconstruction QMF bank.

![Synthesis Filter Bank](image)

**14.31 (a)** The input-output relation of the \( \ell \)–th two-pair is given by
\[
\begin{bmatrix}
R_\ell(z) \\
S_\ell(z)
\end{bmatrix} = \begin{bmatrix}
1 & k_\ell \\
k_\ell & 1
\end{bmatrix} \begin{bmatrix}
R_{\ell-1}(z) \\
S_{\ell-1}(z)
\end{bmatrix},
\]
from which we obtain
\[
R_\ell(z) = R_{\ell-1}(z) + k_\ell[z^{-2}S_{\ell-1}(z)] \quad \text{and} \quad S_\ell(z) = k_\ellR_{\ell-1}(z) + [z^{-2}S_{\ell-1}(z)].
\]
The realization of the two-pair is thus as shown below:

![Realization of Two-Pair](image)

(b) The portion of Figure P14.5 upto the \( m \)-th stage is then as shown below:
Analysis of the above structure yields \( P_m(z) = P_{m-1}(z) + k_m z^{-2} Q_{m-1}(z) \) and \( Q_m(z) = k_m P_{m-1}(z) + z^{-2} Q_{m-1}(z) \).

(c) Analysis of the structure of Figure P14.5 yields \( H_0(z) = P_M(z) + Q_M(z) = P_M(z) + z^{-(2M+1)} P_M(z^{-1}) \). Hence, \( H_0(z^{-1}) = P_M(z^{-1}) + z^{(2M+1)} P_M(z) \). Therefore, \( z^{-(2M+1)} H_0(z^{-1}) = z^{-2(2M+1)} P_M(z^{-1}) + P_M(z) = H_0(z) \). Since the order of \( H_0(z) \) is \( 2M+1 \), \( H_0(z) \) is a Type 2 linear-phase FIR transfer function.

Similarly, analysis of the structure of Figure P14.5 yields \( H_1(z) = P_M(z) - Q_M(z) = P_M(z) - z^{-(2M+1)} P_M(z^{-1}) \). Hence, \( H_1(z^{-1}) = P_M(z^{-1}) - z^{(2M+1)} P_M(z) \). Therefore, \( z^{-(2M+1)} H_1(z^{-1}) = z^{-2(2M+1)} P_M(z^{-1}) - P_M(z) = -H_1(z) \). Since the order of \( H_1(z) \) is \( 2M+1 \), \( H_1(z) \) is a Type 4 linear-phase FIR transfer function.

(c) It follows from Figure P14.5 that
\[
E(z) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ \end{bmatrix} \begin{bmatrix} 1 \\ k_M \\ \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \\ \end{bmatrix} \begin{bmatrix} 1 \\ k_1 \\ \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \\ \end{bmatrix} \begin{bmatrix} 1 \\ k_0 \\ \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \end{bmatrix}
\]
\[
= D_2 \Gamma(z) T_M \ldots T_1 \Gamma(z) T_0,
\]
where \( D_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ \end{bmatrix} \) and \( \Gamma(z) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \\ \end{bmatrix} \). We determine the synthesis bank for perfect reconstruction by choosing \( R(z) = a z^{-M} E^{-1}(z) = a z^{-M} D_2^{-1} \Gamma^{-1}(z) T_M^{-1} \ldots T_1^{-1} \Gamma^{-1}(z) T_0^{-1} \). The final realization of the synthesis filter bank is shown below which satisfies the perfect reconstruction condition within a scale factor.

14.32 From Figure P14.5 we observe that \( H_0(z) = P_M(z) + Q_M(z) \) and \( H_1(z) = P_M(z) - Q_M(z) \). Solving these equations we get
\[
P_M(z) = \frac{1}{2} [H_0(z) + H_1(z)] \quad \text{and} \quad Q_M(z) = \frac{1}{2} [H_0(z) - H_1(z)].
\]
Next, from the
solution of Problem 14.31, Part (b), we have \( P_m(z) = P_{m-1}(z) + k_m z^{-2} Q_{m-1}(z) \)
and \( Q_m(z) = k_m P_{m-1}(z) + z^{-2} Q_{m-1}(z) \). Solving these two equations we get
\[
P_{m-1}(z) = \frac{1}{1-k_m^2} \left[ P_m(z) - k_m Q_m(z) \right]
\quad \text{and} \quad
Q_{m-1}(z) = \frac{1}{(1-k_m^2)z^{-2}} \left[ Q_m(z) - k_m P_m(z) \right].
\]

To synthesize the analysis filter bank, choose \( k_m \) so that the coefficient of the highest power of \( z^{-1} \) in \( P_{m-1}(z) \) is equal to 0. It can be shown that this choice of also guarantees that the coefficient of the second highest power of is also 0. Hence, the order of \( P_{m-1}(z) \) is 2 less that of \( P_m(z) \).

(a) Here \( M = 2 \). Hence, \( P_2(z) = \frac{1}{2} [H_0(z) + H_1(z)] = 1 + 2z^{-1} + 12z^{-2} + 15z^{-3} \)
\( + 4z^{-4} + z^{-5} \) and \( Q_2(z) = \frac{1}{2} [H_0(z) - H_1(z)] = 2 + 4z^{-1} + 15z^{-2} + 12z^{-3} + 2z^{-4} + z^{-5} \).

We form
\[
P_2(z) - k_2 Q_2(z) = (1 - 2k_2) + (2 - 4k_2) z^{-1} + (12 - 15k_2) z^{-2} + (15 - 12k_2) z^{-3}
\quad \text{+} \quad (4 - 2k_2) z^{-4} + (2 - k_2) z^{-5}.
\]
Choose \( k_2 = 2 \). Then
\[
P_2(z) - k_2 Q_2(z) = -3 - 6z^{-1} - 18z^{-2} - 9z^{-3}.
\]
Thus,
\[
P_1(z) = \frac{1}{1-k_2^2} \left[ P_2(z) - k_2 Q_2(z) \right] = 1 + 2z^{-1} + 6z^{-2} + 3z^{-3}.
\]
Likewise,
\[
Q_1(z) = \frac{1}{(1-k_2^2)z^{-2}} \left[ Q_2(z) - k_2 P_2(z) \right] = 3 + 6z^{-1} + 2z^{-2} + z^{-3}.
\]
Next, we form
\[
P_1(z) - k_1 Q_1(z) = (1 - 3k_1) + (2 - 6k_1) z^{-1} + (6 - 2k_1) z^{-2} + (3 - k_1) z^{-3}.
\]
Choose \( k_1 = 3 \). Then,
\[
P_0(z) = \frac{1}{1-k_1^2} \left[ P_1(z) - k_1 Q_1(z) \right] = 1 + 2z^{-1}.
\]
Likewise,
\[
Q_0(z) = \frac{1}{(1-k_1^2)z^{-2}} \left[ Q_1(z) - k_1 P_1(z) \right] = 2 + z^{-1}.
\]
From Figure P14.5 we have
\[
P_0(z) = 1 + k_0 z^{-1} \quad \text{and} \quad Q_0(z) = k_0 + z^{-1}.
\]
Hence, it follows that \( k_0 = 2 \). As a result, the analysis filter bank is as shown below:

The corresponding synthesis filter bank for a perfect reconstruction filter bank is as shown below:
(b) Here $M = 2$. Hence, $P_2(z) = \frac{1}{2} [H_0(z) + H_1(z)] = 1 - 2.5z^{-1} - 13.5z^{-2} + 18z^{-3} + 5z^{-4} - 2z^{-5}$ and
$Q_2(z) = \frac{1}{2} [H_0(z) - H_1(z)] = -2 + 5z^{-1} + 18z^{-2} - 13.5z^{-3} - 2.5z^{-4} + z^{-5}$. We form
$P_2(z) - k_2Q_2(z) = (1 + 2k_2) - (2.5 + 5k_2)z^{-1} - (13.5 + 18k_2)z^{-2} + (18 + 13.5k_2)z^{-3} + (5 + 2.5k_2)z^{-4} - (2 + k_2)z^{-5}$. Choose $k_2 = -2$. Then $P_1(z) = \frac{1}{1 - k_2^2} [P_2(z) - k_2Q_2(z)] = 1 - 2.5z^{-1} - 7.5z^{-2} + 3z^{-3}$. Likewise,
$Q_1(z) = \frac{1}{(1 - k_2^2)z^{-2}} [Q_2(z) - k_2P_2(z)] = 3 - 7.5z^{-1} - 2.5z^{-2} + z^{-3}$. Next, we form
$P_1(z) - k_1Q_1(z) = (1 - 3k_1) - (2.5 - 7.5k_1)z^{-1} - (7.5 - 2.5k_1)z^{-2} + (3 - k_1)z^{-3}$. Choose
$k_1 = 3$. Then, $P_0(z) = \frac{1}{1 - k_1^2} [P_1(z) - k_1Q_1(z)] = 1 - 2.5z^{-1}$. Likewise,
$Q_0(z) = \frac{1}{(1 - k_1^2)z^{-2}} [Q_1(z) - k_1P_1(z)] = -2 + 5z^{-1}$. Hence, $k_0 = -2.5$. As a result, the
analysis filter bank is as shown below:

The corresponding synthesis filter bank for a perfect reconstruction filter bank is as shown below:

(c) Here $M = 2$. Hence, $P_2(z) = \frac{1}{2} [H_0(z) + H_1(z)] = 1 - 2.5z^{-1} + 0.75z^{-2} + 6z^{-3}$
We form

\[
P_2(z) - k_2Q_2(z) = (1 - 2k_2) - (2.5 - 5k_2)z^{-1} - (0.75 - 6k_2)z^{-2} + (6 - 0.75k_2)z^{-3} - (5 - 2.5k_2)z^{-4} + (2 - k_2)z^{-5}.
\]

Choose \(k_2 = 2\). Then

\[
P_1(z) = \frac{1}{1-k_2^2}[P_2(z) - k_2Q_2(z)] = 1 - 2.5z^{-1} + 3.75z^{-2} - 1.5z^{-3}.
\]

Likewise,

\[
Q_1(z) = \frac{1}{(1-k_2^2)z^{-2}}[Q_2(z) - k_2P_2(z)] = -1.5 + 3.75z^{-1} - 2.5z^{-2} + z^{-3}.
\]

Next, we form

\[
P_1(z) - k_1Q_1(z) = (1 + 1.5k_1) - (2.5 - 3.75k_1)z^{-1} + (3.75 + 2.5k_1)z^{-2} - (1.5 + k_1)z^{-3}.
\]

Choose \(k_1 = -1.5\). Then,

\[
P_0(z) = \frac{1}{1-k_1^2}[P_1(z) - k_1Q_1(z)] = 1 - 2.5z^{-1}.
\]

Likewise,

\[
Q_0(z) = \frac{1}{(1-k_1^2)z^{-2}}[Q_1(z) - k_1P_1(z)] = -2.5 + z^{-1}.
\]

Hence, \(k_0 = -2.5\). As a result, the analysis filter bank is as shown below:

The corresponding synthesis filter bank for a perfect reconstruction filter bank is as shown below:

14.33 To show the system of Figure 14.18 is, in general, periodic with a period \(L\), we need to show that if \(\hat{X}_1(z)\) is the output for an input \(X_1(z)\), and \(\hat{X}_2(z)\) is the output for an input \(X_2(z)\), then if \(X_2(z) = z^{-L}X_1(z)\), i.e., \(x_2[n] = x_1[n-L]\), then the corresponding output satisfies \(\hat{X}_2(z) = z^{-L}\hat{X}_1(z)\), i.e., \(\hat{x}_2[n] = \hat{x}_1[n-L]\). Now,

\[
\hat{X}(z) = g^T(z)H^{(m)}(z)\begin{bmatrix} X(z) \\ X(zW) \\ \vdots \\ X(zWL^{-1}) \end{bmatrix},
\]

where \(g(z)\) is the vector composed of the synthesis bank filters given by Eq. (14.98) and \(H^{(m)}(z)\) is the analysis filter bank modulation.
matrix given Eq. (14.102). Now,
\[
\hat{X}_2(z) = g^T(z)H^{(m)}(z) = \begin{bmatrix} X_2(z) \\ X_2(zW) \\ \vdots \\ X_2(zWL^{L-1}) \end{bmatrix}
\]
\[
= g^T(z)H^{(m)}(z) \begin{bmatrix} z^{-L}X_1(z) \\ z^{-L}X_1(zW) \\ \vdots \\ z^{-L}X_1(zWL^{L-1}) \end{bmatrix} = z^{-L}g^T(z)H^{(m)}(z) \begin{bmatrix} X_1(z) \\ X_1(zW) \\ \vdots \\ X_1(zWL^{L-1}) \end{bmatrix} = z^{-L}\hat{X}_1(z).
\]
As a result, the structure of Figure 14.18 is, in general, a time-varying system with a period of \(L\).

14.34 \(H_0(z) = E_{00}(z^2) + z^{-1}E_{01}(z^2), H_1(z) = E_{10}(z^2) + z^{-1}E_{11}(z^2)\). Thus,
\[
E(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) \\ E_{10}(z) & E_{11}(z) \end{bmatrix}
\]
\[(a) \quad H_0(z) = E_0(z^2) + z^{-1}E_1(z^2). \text{ Hence, } H_1(z) = H_0(-z) = E_0(z^2) - z^{-1}E_1(z^2).
\]
Thus, \(E_{00}(z) = E_0(z), E_{01}(z) = E_1(z), E_{10}(z) = E_0(z), \text{ and } E_{11}(z) = -E_1(z)\).

Therefore,
\[
E(z) = \begin{bmatrix} E_0(z) & E_1(z) \\ E_0(z) & -E_1(z) \end{bmatrix}
\]
\[(b) \quad \text{Two cases need to be considered – } N\text{ odd and } N\text{ even. Consider first the } N\text{ odd case.}
\]
For simplicity let \(H_0(z) = a_0 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3} + a_4z^{-4}\). Then,
\[
H_1(z) = z^{-3}H_0(z^{-1}) = z^{-3}(a_0 + a_1z + a_2z^2 + a_3z^3) = a_3 + a_2z^{-1} + a_1z^{-2} + a_0z^{-3}.
\]
Thus, \(E_{00}(z) = E_0(z) = a_0 + a_2z^{-1}, E_{01}(z) = E_1(z) = a_1 + a_3z^{-1}, \text{ and } E_{10}(z) = a_3 + a_1z^{-1}, E_{11}(z) = a_2 + a_0z^{-1} \Rightarrow E_{10}(z) = z^{-1}E_1(z^{-1}) \text{ and } E_{11}(z) = z^{-1}E_0(z^{-1})\).

Next, consider the \(N\) even case. Assume \(H_0(z) = a_0 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3} + a_4z^{-4}\).

Then,
\[
H_1(z) = z^{-4}H_0(z^{-1}) = z^{-4}(a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4) = a_4 + a_3z^{-1} + a_2z^{-2} + a_1z^{-3} + a_0z^{-4}.
\]
Thus,
\[
E_{00}(z) = E_0(z) = a_0 + a_2z^{-1} + a_4z^{-2}, E_{01}(z) = E_1(z) = a_1 + a_3z^{-1}, \text{ and } E_{10}(z) = a_4 + a_2z^{-1} + a_0z^{-2}, E_{11}(z) = a_3 + a_1z^{-1} \Rightarrow E_{10}(z) = z^{-2}E_0(z^{-1}) \text{ and } E_{11}(z) = z^{-1}E_1(z^{-1})\).

In the general case, for \(N\) odd with \(N = 2K + 1\), \(E_{11}(z) = z^{-K}E_0(z^{-1})\) and
\[
E_{10}(z) = z^{-K}E_1(z^{-1}) \Rightarrow E(z) = \begin{bmatrix} E_0(z) & E_1(z) \\ z^{-(N-1)/2}E_1(z^{-1}) & z^{-(N-1)/2}E_0(z^{-1}) \end{bmatrix}.
\]

Not for sale
Likewise, for $N$ even with $N = 2K$, $E_{10}(z) = z^{-K}E_0(z^{-1})$ and

$$E_{11}(z) = z^{-(K-1)}E_1(z^{-1}) \Rightarrow E(z) = \begin{bmatrix} E_0(z) & E_1(z) \\ z^{-N/2}E_0(z^{-1}) & z^{-(N-2)/2}E_1(z^{-1}) \end{bmatrix}.$$  

14.35 (a) $H_0(z) = \frac{1}{4} [A_0(z^2) + z^{-1}A_1(z^2)]$ and $H_1(z) = \frac{1}{4} [A_0(z^2) - z^{-1}A_1(z^2)].$

Hence, $E(z) = \begin{bmatrix} A_0(z) & A_1(z) \\ A_0(z) & -A_1(z) \end{bmatrix}$, and $R(z) = \begin{bmatrix} A_1(z) & A_1(z) \\ A_0(z) & -A_0(z) \end{bmatrix}.$

(b) To prove $E(z)$ is lossless, we need to show $E^\dagger(e^{j\omega})E(e^{j\omega}) = c^2 I$ for some constant $c$: $E^\dagger(e^{j\omega})E(e^{j\omega}) = \begin{bmatrix} A_0^\dagger(e^{j\omega}) & A_1^\dagger(e^{j\omega}) \\ A_1(e^{j\omega}) & -A_1(e^{j\omega}) \end{bmatrix} \begin{bmatrix} A_0(e^{j\omega}) & A_1(e^{j\omega}) \\ A_0(e^{j\omega}) & -A_1(e^{j\omega}) \end{bmatrix} = \begin{bmatrix} |A_0(e^{j\omega})|^2 + |A_1(e^{j\omega})|^2 & 0 \\ 0 & |A_1(e^{j\omega})|^2 + |A_1(e^{j\omega})|^2 \end{bmatrix} = 2I$. Hence, $E(z)$ is lossless.

(c) $R(z)E(z) = \frac{1}{4} \begin{bmatrix} 2A_1(z)A_0(z) & 0 \\ 0 & 2A_1(z)A_0(z) \end{bmatrix} = \frac{A_1(z)A_0(z)}{2} I$. Therefore, $R(z) = \frac{A_1(z)A_0(z)}{2} E^{-1}(z)$.

(d) As in part (c), $R(z)E(z) = \frac{A_1(z)A_0(z)}{2} I$.

14.36 $E(z) = \begin{bmatrix} E_{00}(z) & E_{01}(z) \\ E_{10}(z) & E_{11}(z) \end{bmatrix} = \begin{bmatrix} -2z^{-2} + 8z^{-1} - 6 & 2 - z^{-1} \\ -3 + 2z^{-1} & 1 \end{bmatrix}$. Hence, $H_0(z) = E_{00}(z^2) + z^{-1}E_{01}(z^2) = -6 + 2z^{-1} + 8z^{-2} - z^{-3} - 2z^{-4}$ and $H_1(z) = E_{10}(z^2) + z^{-1}E_{11}(z^2) = -3 + z^{-1} + 2z^{-2}$.

Now, det$(E(z)) = (-2z^{-2} + 8z^{-1} - 6) - (2 - z^{-1})(-3 + 2z^{-1}) = z^{-1}$. From Eq. (14.133) we have

$$R(z) = \begin{bmatrix} R_{00}(z) & R_{01}(z) \\ R_{10}(z) & R_{11}(z) \end{bmatrix} = cz^{-K}E_1^{-1}(z) = \frac{cz^{-K}}{z^{-1}} \begin{bmatrix} -2z^{-2} + 8z^{-1} - 6 & -2 + z^{-1} \\ 3 - 2z^{-1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2z^{-2} + 8z^{-1} - 6 & -2 + z^{-1} \\ 3 - 2z^{-1} & 1 \end{bmatrix}$$

for $K = 1$ and $c = 1$. Therefore, $G_0(z) = R_{00}(z^2) + z^{-1}R_{01}(z^2) = -6 - 2z^{-1} + 8z^{-2} + z^{-3} - 2z^{-4}$ and $G_1(z) = R_{10}(z^2) + z^{-1}R_{11}(z^2) = 3 + z^{-1} - 2z^{-2}$.
From Eqs. (14.121) and (14.122a) to (14.122c), we thus get

\[
H_0(z) = 2 - 2z^{-1} + z^{-2}, H_1(z) = 3 + z^{-1} - 2z^{-2}, H_2(z) = 2 + 4z^{-1} + 2z^{-2}.
\]

Now, \(4P^{-1} = \begin{bmatrix}
0.8 & 0.64 & 0.24 \\
-0.8 & 0.16 & 0.56 \\
0.8 & -0.96 & 0.64
\end{bmatrix}\). Using Eqs. (14.124) and (14.125a) to (14.125c), we arrive at

\[
\begin{bmatrix}
G_0(z) \\
G_1(z) \\
G_2(z)
\end{bmatrix} = \begin{bmatrix} z^{-2} & z^{-1} & 1 \end{bmatrix} \begin{bmatrix}
0.8 & 0.64 & 0.24 \\
-0.8 & 0.16 & 0.56 \\
0.8 & -0.96 & 0.64
\end{bmatrix},
\]

leading to the synthesis filters

\[
G_0(z) = 0.8 - 0.8z^{-1} + 0.8z^{-2}, G_1(z) = -0.96 + 0.16z^{-1} + 0.64z^{-2},
G_2(z) = 0.64 + 0.56z^{-1} + 0.24z^{-2}.
\]

For \(y[n] = 4x[n - 3]\) to hold, we require \(R(z)E(z) = 4I\). Hence, \(E(z) = 4[R(z)]^{-1} = 4 \begin{bmatrix}
1 & -2 & 3 & -1 \\
2 & 1 & 1 & 0 \\
0 & 3 & -1 & 1 \\
1 & 1 & -1 & 2
\end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix}
-1.4118 & 2.5882 & -1.8824 & 0.2353 \\
0.2353 & 0.2353 & 1.6471 & -0.7059 \\
2.5882 & -1.4118 & 2.1176 & 0.2353 \\
1.8824 & -2.1176 & 1.1765 & 2.3529
\end{bmatrix}.
\]

Therefore,

\[
H_0(z)H_1(z)H_2(z) = \begin{bmatrix} z^{-2} & z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\
3 & -3 & 1 \\
2 & 1 & 2
\end{bmatrix}.
\]

For perfect reconstruction we require the Type 2 polyphase component matrix to be given

\[
R(z) = [E(z)]^{-1} = \begin{bmatrix}
8/3 & 5/3 & -3 \\
-2/3 & -2/3 & 1 \\
-8/3 & 4/3 & 3
\end{bmatrix}.
\]

Hence, the synthesis filters are obtained from

\[
H_0(z)H_1(z)H_2(z) = \begin{bmatrix} z^{-2} & z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\
1 & -3 & 2 \\
2 & 1 & 2
\end{bmatrix}.
\]
\[ \begin{bmatrix} G_0(z) & G_1(z) & G_2(z) \end{bmatrix} = \begin{bmatrix} z^{-2} & z^{-1} & 1 \\ \begin{bmatrix} 8/3 & 5/3 & -3 \\ -2/3 & -2/3 & 1 \\ -8/3 & -4/3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ \end{bmatrix} \] resulting in

\[ G_0(z) = -\frac{8}{3} - \frac{2}{3} z^{-1} + \frac{8}{3} z^{-2}, \quad G_1(z) = \frac{4}{3} - \frac{2}{3} z^{-1} + \frac{5}{3} z^{-2}, \quad \text{and} \quad G_2(z) = 3 + z^{-1} + 3z^{-2}. \]

14.40 \( E(z) = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 3 & 1 & -1 & -2 \\ 3 & 1 & 2 & -1 \\ 1 & 3 & 1 & 2 \end{bmatrix} \). For perfect reconstruction we require

\[ R(z) = E^{-1}(z) = \frac{1}{3} \begin{bmatrix} 1 & 0.5 & 0 & -0.5 \\ -1 & 0.25 & 0 & 1.25 \\ -0.4 & -0.95 & 1.2 & 0.05 \\ 1.2 & -0.15 & -0.6 & -0.15 \end{bmatrix}. \]

Thus,

\[ \begin{bmatrix} G_0(z) & G_1(z) & G_2(z) & G_3(z) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} z^{-3} & z^{-2} & z^{-1} & 1 \\ \begin{bmatrix} 1 \\ \end{bmatrix} & \begin{bmatrix} 1 & 0.5 & 0 & -0.5 \\ \end{bmatrix} & \begin{bmatrix} -1 \\ \end{bmatrix} & \begin{bmatrix} -0.4 & -0.95 & 1.2 & 0.05 \\ \end{bmatrix} & \begin{bmatrix} 1.2 & -0.15 & -0.6 & -0.15 \end{bmatrix} \end{bmatrix} \]

Hence,

\[ G_0(z) = \frac{1}{3}(1.2 - 0.4z^{-1} - z^{-2} + z^{-3}), \quad G_1(z) = \frac{1}{3}(-0.15 - 0.95z^{-1} + 0.25z^{-2} + 0.5z^{-3}), \]

\[ G_2(z) = \frac{1}{3}(-0.6 + 1.2z^{-1}), \quad G_3(z) = \frac{1}{3}(-0.15 + 0.05z^{-1} + 1.25z^{-2} - 0.5z^{-3}). \]

14.41 \( G_0(z) = -2 + 3z^{-1} + 2z^{-2}, \quad G_0(z) = 3 + z^{-1} - 2z^{-2}, \quad G_0(z) = 2 - z^{-1} + z^{-2}. \)

Rewriting these equations in an matrix form we get

\[ \begin{bmatrix} G_0(z) & G_1(z) & G_2(z) \end{bmatrix} = \begin{bmatrix} z^{-2} & z^{-1} & 1 \\ \begin{bmatrix} 2 \\ \begin{bmatrix} 2 & -2 & 2 \\ \end{bmatrix} \end{bmatrix} & \begin{bmatrix} 3 \\ \end{bmatrix} & \begin{bmatrix} 1 \\ \end{bmatrix} & \begin{bmatrix} -1 \\ \end{bmatrix} \end{bmatrix} \]

This implies,

\[ R(z) = \begin{bmatrix} 2 & -2 & 2 \\ 3 & 1 & -1 \\ -2 & 3 & 1 \end{bmatrix}. \]

For perfect reconstruction we require

\[ E(z) = R^{-1}(z) = \begin{bmatrix} 0.125 & 0.25 & 0 \\ -0.0313 & 0.1875 & 0.25 \\ 0.3438 & -0.0625 & 0.25 \end{bmatrix}. \]

Hence,

\[ \begin{bmatrix} H_0(z) \\ H_1(z) \\ H_2(z) \end{bmatrix} = \begin{bmatrix} 0.125 & 0.25 & 0 \\ -0.0313 & 0.1875 & 0.25 \\ 0.3438 & -0.0625 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \end{bmatrix} \]

implying

\[ H_0(z) = 0.125 + 0.25z^{-1}, \quad H_1(z) = -0.0313 + 0.1875z^{-1} + 0.25z^{-2}, \]

\[ H_1(z) = 0.3438 - 0.0625z^{-1} + 0.25z^{-2}. \]

14.42 If the filter bank is alias-free, then
The above \( L \) equations are

\[
\begin{bmatrix}
H_0(z) & H_1(z) & \ldots & H_{L-1}(z) \\
H_0(zW) & H_1(zW) & \ldots & H_{L-1}(zW) \\
\vdots & \vdots & \ddots & \vdots \\
H_0(zWL^{-1}) & H_1(zWL^{-1}) & \ldots & H_{L-1}(zWL^{-1})
\end{bmatrix}
\begin{bmatrix}
G_0(z) \\
G_1(z) \\
\vdots \\
G_{L-1}(z)
\end{bmatrix}
= 
\begin{bmatrix}
T(z) \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

The above equations are

(1): \( H_0(z)G_0(z) + H_1(z)G_1(z) + \ldots + H_{L-1}(z)G_{L-1}(z) = T(z), \)

(2): \( H_0(zW^r)G_0(z) + H_1(zW^r)G_1(z) + \ldots + H_{L-1}(zW^r)G_{L-1}(z) = 0, \ 1 \leq r \leq L - 1. \)

Replacing \( z \) by \( zW^{L-r} \) in the above equation we get

\[
H_0(zW^L)G_0(zW^{L-r}) + H_1(zW^L)G_1(zW^{L-r}) + \ldots + H_{L-1}(zW^L)G_{L-1}(zW^{L-r}) = 0,
\]

or equivalently,

(3): \( H_0(z)G_0(zW^{L-r}) + H_1(z)G_1(zW^{L-r}) + \ldots + H_{L-1}(z)G_{L-1}(zW^{L-r}) = 0, \ 0 \leq r \leq L - 1. \)

Rewriting Eqs. (1) and (3) in matrix form we get

\[
\begin{bmatrix}
G_0(z) & G_1(z) & \ldots & G_{L-1}(z) \\
G_0(zW) & G_1(zW) & \ldots & G_{L-1}(zW) \\
\vdots & \vdots & \ddots & \vdots \\
G_0(zWL^{-1}) & G_1(zWL^{-1}) & \ldots & G_{L-1}(zWL^{-1})
\end{bmatrix}
\begin{bmatrix}
H_0(z) \\
H_1(z) \\
\vdots \\
H_{L-1}(z)
\end{bmatrix}
= 
\begin{bmatrix}
T(z) \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

Thus, if an \( L \)-channel filter bank is alias-free with a given set of analysis and synthesis filters, then the filter bank is still alias-free if the analysis and synthesis filters are interchanged.

### 14.43

An equivalent representation of the structure of Figure P14.6 is shown below

where \( \Gamma_m = 
\begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & z^{-1}
\end{bmatrix}
\)

The corresponding synthesis bank for designing an \( L \)-channel perfect reconstruction bank is thus as shown below where \( T_m = S_m^{-1} \) and

\[
\Lambda_m = z^{-2} \Gamma_m^{-1} = 
\begin{bmatrix}
z^{-1} & 0 & \ldots & 0 \\
0 & z^{-1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
H_0(z) \\
H_1(z) \\
H_2(z)
\end{bmatrix} =
\begin{bmatrix}
1 & -2 & 3 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & z^{-1}
\end{bmatrix}
\begin{bmatrix}
3 & -5 & 3 \\
-1 & 3 & -2 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
z^{-1} \\
z^{-1} \\
z^{-2}
\end{bmatrix}
\]

Hence, \( H_0(z) = 5 - 11z^{-1} + 4z^{-2} + 3z^{-3} \), \( H_1(z) = -1 + 3z^{-1} - 2z^{-3} \), \( H_2(z) = -z^{-2} + z^{-3} \).

\[
\begin{bmatrix}
G_0(z) & G_1(z) & G_2(z)
\end{bmatrix} =
\begin{bmatrix}
z^{-2} & z^{-1} & 1 \\
z^{-1} & 0 & 0 \\
z^{-1} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & -2 & 3 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
z^{-1} & 0 & 0 \\
z^{-1} & 0 & 0 \\
z^{-1} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
3 & -5 & 3 \\
-1 & 3 & -2 \\
0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
1 \\
z^{-1} \\
z^{-1}
\end{bmatrix}
\begin{bmatrix}
3z^{-1} + 1 & 8z^{-1} + 3 & 7z^{-1} + 4 \\
z^{-1} + 2 & 3z^{-1} + 6 & 3z^{-1} + 8 \\
1 & 3 & 4
\end{bmatrix}
\]

Hence, \( G_0(z) = 1 + 2z^{-1} + 2z^{-2} + 3z^{-3} \), \( G_1(z) = 3 + 6z^{-1} + 6z^{-2} + 8z^{-3} \), \( G_2(z) = 4 + 8z^{-1} + 7z^{-2} + 7z^{-3} \).

**14.45** Figure 14.20 with internal variables labeled is shown below:
From the above figure it follows that we can express the z-transforms of \( \{c_\ell[n]\} \) as
\[
C_\ell(z) = \frac{1}{L} \sum_{\ell=0}^{L-1} (z^{1/L}W^k)^{-\ell} X((z^{1/L}W^k), \ 0 \leq \ell \leq L-1, \ \text{where} \ W = e^{-j 2\pi / L}.
\]
Likewise, the z-transforms of \( \{b_s[n]\} \) can be expressed as
\[
B_s(z) = \sum_{s=0}^{L-1} P_{s,\ell}(z)C_\ell(z), \ 0 \leq s \leq L-1, \ \text{where} \ P_{s,\ell}(z) \ \text{denotes the z-transforms the (s, \ell) -th element of} \ \mathbf{P}(z).
\]
Finally, the z-transform of the output \( y[n] \) is given by
\[
Y(z) = \sum_{s=0}^{L-1} z^{-(L-1-s)}B_s(z^L) = \sum_{s=0}^{L-1} z^{-(L-1-s)} \sum_{\ell=0}^{L-1} P_{s,\ell}(z^L)C_\ell(z^L)
\]
\[
= \frac{1}{L} \sum_{s=0}^{L-1} z^{-(L-1-s)} \sum_{\ell=0}^{L-1} P_{s,\ell}(z^L) \sum_{k=0}^{L-1} (zW^k)^{-\ell} X(zW^k)
\]
\[
= \frac{1}{L} \sum_{k=0}^{L-1} X(zW^k) \sum_{\ell=0}^{L-1} z^{-\ell} P_{s,\ell}(z^L).
\]
In the above expression, terms of the form \( X(zW^k), k \neq 0, \) represent the contribution coming from aliasing. Hence, the expression for \( Y(z) \) is free from these aliasing terms for any arbitrary input \( x[n] \) if and only if
\[
\sum_{\ell=0}^{L-1} W^{-k\ell} \sum_{s=0}^{L-1} z^{-(L-1-s)}z^{-\ell} P_{s,\ell}(z^L) = 0, \ k \neq 0.
\]
The above expression can be written in a matrix form as
\[
\mathbf{D}^{\dagger} \begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{L-1}(z) \end{bmatrix} = \begin{bmatrix} T(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix},
\]
where \( \mathbf{D} \) is the DFT matrix, \( T(z) \) is transfer function of the alias-free system, and
\[
V_\ell(z) = \sum_{s=0}^{L-1} z^{-\ell} z^{-(L-1-s)} P_{s,\ell}(z^L). \ \text{Since} \ \mathbf{D}^{\dagger} \mathbf{D} = L \mathbf{I}, \ \text{the above matrix equation can be alternately written as}
\]
\[
\begin{bmatrix} V_0(z) \\ V_1(z) \\ \vdots \\ V_{L-1}(z) \end{bmatrix} = \mathbf{D} \begin{bmatrix} T(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \ \text{This implies that}
\]
\[
V_\ell(z) = V(z), \ 0 \leq \ell \leq L-1, \ \text{as the first column of} \ \mathbf{D} \ \text{has all elements equal to 1.} \ \text{As a result, the} \ L \text{-channel QMF bank is alias-free if and only if} \ V_\ell(z) \ \text{is the same for all} \ \ell.
\]
The two figures below show the polyphase realizations of \( V_0(z) \) and \( V_1(z) : \)
The realization of $V_1(z)$ can be redrawn as indicated below:

Because of the constraint $V_0(z) = V_1(z)$, the polyphase components in Figures (a) and (c) should be the same. From these two figures it follows that the first column of $P(z)$ is an upwards-shifted version of the second column, with the topmost element appearing with a $z^{-1}$ attached. This type of relation holds for the $k$-th column and the $(k+1)$-th column of $P(z)$. As a result, $P(z)$ is a pseudo-circulant matrix of the form of Eq. (14.155).

**14.46** $H_0(z) = E_0(z^2) + z^{-1}E_1(z^2)$, $H_1(z) = E_0(z^2) - z^{-1}E_1(z^2)$. Hence,

$$
E(z) = \begin{bmatrix} E_0(z) & E_1(z) \\ E_0(z) & -E_1(z) \end{bmatrix}.
$$
Likewise,

$$
G_0(z) = E_0(z^2) + z^{-1}E_1(z^2) = z^{-1}R_{00}(z^2) + R_{10}(z^2),
$$
$$
G_1(z) = -E_0(z^2) + z^{-1}E_1(z^2) = z^{-1}R_{01}(z^2) + R_{11}(z^2).
$$
Hence,

$$
R(z) = \begin{bmatrix} R_{00}(z) & R_{01}(z) \\ R_{10}(z) & R_{11}(z) \end{bmatrix} = \begin{bmatrix} E_1(z) & E_1(z) \\ E_0(z) & -E_0(z) \end{bmatrix}.
$$
Therefore,
\[
P(z) = R(z)E(z) = \begin{bmatrix} E_1(z) & E_1(z) \\ E_0(z) & -E_0(z) \end{bmatrix} \begin{bmatrix} E_0(z) & E_1(z) \\ E_0(z) & -E_1(z) \end{bmatrix} = \begin{bmatrix} 2E_0(z)E_1(z) & 0 \\ 0 & 2E_0(z)E_1(z) \end{bmatrix}
\]

Here, \( P_0(z) = 2E_0(z)E_1(z) \) and \( P_1(z) = 0 \). As a result, \( P(z) \) is pseudo-circulant.

14.47 (a) \( H_0(z) = 1 - 2z^{-1} + 4.5z^{-2} + 6z^{-3} + z^{-4} + 0.5z^{-5} \),
\( H_0(z^{-1}) = 0.5z^5 + z^4 + 6z^3 + 4.5z^2 - 2z + 1 \). Therefore,
\( H_0(z)H_0(z^{-1}) = 0.5z^5 + 6.25z^3 + 22.5z + 62.5 + 22.5z^{-1} + 6.25z^{-3} + 0.5z^{-5} \). Thus,
\( H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 125 \). Hence, \( H_0(z) \) is a power-symmetric function.

The highpass analysis filter is given by
\( H_1(z) = z^{-5}H_0(-z^{-1}) = -0.5 + z^{-1} - 6z^{-2} + 4.5z^{-3} + 2z^{-4} + z^{-5} \). The two synthesis filters are time-reversed versions of the analysis filters as per Eq. (14.92):
\( G_0(z) = 2z^{-5}H_0(z^{-1}) = 1 + 2z^{-1} + 12z^{-2} + 9z^{-3} - 4z^{-4} + 2z^{-5} \) and
\( G_1(z) = 2z^{-5}H_1(z^{-1}) = 2 + 4z^{-1} + 9z^{-2} - 12z^{-3} + 2z^{-4} - z^{-5} \).

(b) \( H_0(z) = 1 + \frac{1}{2}z^{-1} + \frac{15}{4}z^{-2} - z^{-4} + 2z^{-5}, H_0(z^{-1}) = 2z^5 - z^4 + \frac{15}{4}z^2 + \frac{1}{2}z + 1 \).

Therefore,
\( H_0(z)H_0(z^{-1}) = 2z^5 + 10.75z^3 - 3.375z + 20.3125 - 3.375z^{-1} + 10.75z^{-3} + 2z^{-5} \). Thus,
\( H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 40.625 \). Hence, \( H_0(z) \) is a power-symmetric function.

The highpass analysis filter is given by
\( H_1(z) = z^{-5}H_0(-z^{-1}) = -2 - z^{-1} + \frac{15}{4}z^{-3} - \frac{1}{2}z^{-4} + z^{-5} \). The two synthesis filters are time-reversed versions of the analysis filters as per Eq. (14.92):
\( G_0(z) = 2z^{-5}H_0(z^{-1}) = 4 - 2z^{-1} + \frac{15}{2}z^{-3} + z^{-4} + 2z^{-5} \) and
\( G_1(z) = 2z^{-5}H_1(z^{-1}) = 2 - z^{-1} + \frac{15}{2}z^{-2} - 2z^{-4} - 4z^{-5} \).

14.48 \( H_0(z) = 1 + az^{-1} + z^{-2} \) and \( H_1(z) = 1 + az^{-1} + bz^{-2} + az^{-3} + z^{-4} \). The corresponding synthesis filters are given by \( G_0(z) = H_1(-z) = 1 - az^{-1} + bz^{-2} - az^{-3} + z^{-4} \), and \( G_1(z) = -H_0(-z) = -1 + az^{-1} - z^{-2} \).

To show that the filter bank is alias-free and satisfies the perfect reconstruction property we need to show that
\[
\begin{bmatrix} H_0(z) & H_0(-z) \\ H_0(-z) & H_1(-z) \end{bmatrix} \begin{bmatrix} G_0(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} cz^{-K} \\ 0 \end{bmatrix}, \text{ where } c \neq 0.
\]

Thus, if \( a \neq 0 \) and \( b \neq 0 \)
non the above filter bank is alias-free and also satisfies the perfect reconstruction property.

14.49 If \( H_0(z) \) is required to have 2 zeros at \( z = -1 \), then it is of the form

\[
H_0(z) = (1 + z^{-1})^2 C(z),
\]

where \( C(z) \) is a first-order polynomial. Now,

\[
P(z) = H_0(z)H_0(z^{-1}) = (1 + z^{-1})^2 (1 + z)^2 R(z),
\]

where \( R(z) \) is a zero-phase polynomial of the form \( R(z) = az + b + az^{-1} \). For perfect reconstruction we require, \( P(z) + P(-z) = 2 \), i.e.,

\[
(1 + z^{-1})^2 (1 + z)^2 (az + b + az^{-1}) + (1 - z^{-1})^2 (1 - z)^2 (-az + b - az^{-1}) = 2. \]

Since the above equation must hold for all \( z \), we observe that at \( z = 1 \), we get

\[
2a + b = \frac{1}{8}. \]

Likewise, at \( z = j \), we get \( b = \frac{1}{4} \). Hence, \( a = -\frac{1}{16} \). Therefore,

\[
P(z) = (1 + z^{-1})^2 (1 + z)^2 (-\frac{1}{16} z + \frac{1}{4} - \frac{1}{16} z^{-1}). \]

The analysis filter \( H_0(z) \) is obtained by a spectral factorization of \( P(z) \). Three choices of spectral factorization resulting in linear-phase analysis filters are given in Section 14.3.3.

14.50

14.51 (a) An orthogonal perfect reconstruction filter bank is obtained by choosing the minimum-phase spectral factor of \( P(z) \) as the lowpass analysis filter \( H_0(z) \) and then determining the lowpass synthesis filter \( G_0(z) \) using the relation

\[
G_0(z) = z^{-5} H_0(z^{-1}). \]

The minimum-phase spectral factor of \( P(z) \) is given by

\[
H_0(z) = 0.33267(1 + z^{-1})^3 (1 - 0.5745z^{-1} + 0.1059z^{-2}). \]

Hence,

\[
G_0(z) = z^{-5} H_0(z^{-1}) = 0.33267(1 + z^{-1})^3 (0.1059 - 0.5745z^{-1} + z^{-2}). \]

(b) A perfect reconstruction filter bank with symmetric lowpass analysis and synthesis filters of even length is given by

\[
H_0(z) = (1 + z^{-1})(a + bz^{-1} + cz^{-2} + bz^{-3} + az^{-4}) \quad \text{and} \quad G_0(z) = (1 + z^{-1})^5. \]

Another possible design of a perfect reconstruction filter bank with symmetric lowpass analysis and synthesis filters of even length is given by

\[
H_0(z) = (1 + z^{-1})^3(a + bz^{-1} + cz^{-2} + bz^{-3} + az^{-4}) \quad \text{and} \quad G_0(z) = (1 + z^{-1})^3. \]

(c) A perfect reconstruction filter bank with symmetric lowpass analysis and synthesis filters of odd length is given by

\[
H_0(z) = (a + bz^{-1} + cz^{-2} + bz^{-3} + az^{-4}) \quad \text{and} \quad G_0(z) = (1 + z^{-1})^6. \]

Another possible design of a perfect reconstruction filter bank with symmetric lowpass analysis and synthesis filters of odd length is given by

\[
H_0(z) = (1 + z^{-1})^2(a + bz^{-1} + cz^{-2} + bz^{-3} + az^{-4}) \quad \text{and} \quad G_0(z) = (1 + z^{-1})^4. \]
The polyphase matrices for the structure of Figure P14.7(a) are given by
\[ E(z) = \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix} \quad \text{and} \quad R(z) = \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix}. \]
Therefore,
\[ R(z)E(z) = \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]
Hence, the structure of Figure P14.7(a) is a perfect reconstruction QMF bank.

The polyphase matrices for the structure of Figure P14.7(b) are given by
\[ E(z) = \begin{bmatrix} 1 & 0 \\ Q(z) & 1 \end{bmatrix} \quad \text{and} \quad R(z) = \begin{bmatrix} 1 & 0 \\ -P(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]
\[ \begin{bmatrix} 1 & 0 \\ -Q(z) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ P(z) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]
Hence, the structure of Figure P14.7(b) is a perfect reconstruction QMF bank.

If the 2-channel QMF banks in the middle of the structure of Figure 14.24 are of perfect reconstruction type, then each of these two filter banks have a distortion transfer function of the form \( \alpha z^{-M} \), where \( M \) is a positive integer.

Likewise, the 2-channel analysis filter bank on the left with the 2-channel synthesis filter bank on the right form a perfect reconstruction QMF bank with a distortion transfer function \( \beta z^{-L} \), where \( L \) is a positive integer:

Hence, an equivalent representation of Figure 14.24 is as indicated below:
which reduces to

\[
\begin{align*}
    \alpha z^{-M} & \quad \beta z^{-L} \\
    H_L(z) & \downarrow 2 & \alpha z^{-M} & \uparrow 2 & \beta z^{-L} \\
    H_H(z) & \downarrow 2 & \alpha z^{-M} & \uparrow 2 & \beta z^{-L} \\
    \alpha z^{-2M} & \quad \beta z^{-L} \\
\end{align*}
\]

Thus, the overall structure is also of perfect reconstruction type with a distortion transfer function given by \( \alpha \beta z^{-(2M+L)} \).

14.57 We analyze the 3-channel filter bank of Figure 14.27(b). If the 2-channel QMF bank of Figure 14.27(a) is of perfect reconstruction type with a distortion transfer function \( \beta z^{-L} \), the structure of Figure 14.27(b) should be implemented as indicated below to ensure perfect reconstruction:

An equivalent representation of the above structure is as shown below:

which reduces to

\[
\begin{align*}
    \beta z^{-2L} & \quad \beta z^{-L} \\
    x[n] & \downarrow 2 & \beta z^{-L} & \uparrow 2 & \beta z^{-L} \\
    H_0(z) & \downarrow 2 & \beta z^{-L} & \uparrow 2 & \beta z^{-L} \\
    H_1(z) & \downarrow 2 & \beta z^{-L} & \uparrow 2 & \beta z^{-L} \\
    \beta z^{-2L} & \quad \beta z^{-L} \\
\end{align*}
\]

verifying the perfect reconstruction property.

In a similar manner, the perfect reconstruction property of Figure 14.27(c) can be proved.
The specifications of the corresponding zero-phase half-band filter are as follows: stopband edge $\omega_s = 0.65\pi$ and a minimum stopband attenuation of $\alpha_s = 2 \times 30 + 6.02 = 66.02$ dB. The desired stopband ripple is therefore $\delta_s = 10^{-\alpha_s / 20} = 10^{-3.301} = 0.0005$. The passband edge of the half-band filter is at $\omega_p = \pi - 0.6\pi = 0.4\pi$. Using the function `remezord` we then estimate the order of $F(z)$ and using the function `remez` we next design $Q(z)$. To this end the code fragments used are

```matlab
[N, fpts, mag, wt] = remezord([0.4, 0.6], [1, 0], [0.0005, 0.005]);
```

The order of $F(z)$ is found to be 30 which is of the form $4K + 2 = N$ for $K = 4$. The order of $H_0(z)$ is therefore 15. The filter $Q(z)$ is designed using the statement

```matlab
[q, err] = remez(N, fpts, mag, wt);
```

To determine the coefficients $\{f[n]\}$ of the filter $F(z)$ we add $\text{err}$ to the central coefficient $q[16]$. Next using the statement $h0 = \text{firminphase}(f)$; we determine the minimum-phase spectral factor of $F(z)$ which are the coefficients of the lowpass analysis filter $H_0(z)$:

<table>
<thead>
<tr>
<th>Columns 1 through 7</th>
<th>0.2818</th>
<th>0.5076</th>
<th>0.3582</th>
<th>-0.0386</th>
<th>-0.1749</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0083</td>
<td>0.1047</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Columns 8 through 14</td>
<td>-0.0089</td>
<td>-0.0663</td>
<td>0.0122</td>
<td>0.0418</td>
<td>-0.0134</td>
</tr>
<tr>
<td>0.0319</td>
<td>0.0335</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Columns 15 through 16</td>
<td>-0.0121</td>
<td>0.0008</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The highpass analysis filter $H_1(z)$ is obtained using the code fragments

```matlab
k = 0:15;
h1 = ((-1).^k).*h0;
```

The synthesis filters $G_0(z)$ and $G_1(z)$ can be found from the analysis filters using the code fragments

```matlab
G0 = fliplr(h0);
G1 = fliplr(h1);
```

Plots of the roots of and the gain responses of the two analysis filters are shown below:
% Program for the design of a two-channel QMF lattice filter bank.
Len = input('The length of the filter = ');
if (mod(Len,2) ~= 0)
    sprintf('Length has to be an even number')
    Len = Len+1;
end
ord = Len/2-1;
ws = 0.55*pi;
kinit = [1;zeros([ord,1])];
% set the parameters for the optimization routine
options = optimset('MaxIter', 2500, 'Display', 'off');
kfin = fminunc('filtopt',kinit,options,Len,ws);
e00old = 1;
e01old = kfin(1);
e10old = -kfin(1);
e11old = 1;
for k = 2:length(kfin)
    e00new = [e00old 0]-kfin(k)*[0 e01old];
e01new = kfin(k)*[e00old 0]+[0 e01old];
e10new = [e10old 0]-kfin(k)*[0 e11old];
e11new = kfin(k)*[e10old 0]+[0 e11old];
e00old = e00new;
e01old = e01new;
e10old = e10new;
e11old = e11new;
end
E1 = [e00old;e01old];
h0 = E1(:);
scale_factor = abs(sum(h0));
h0 = h0/scale_factor;
E2 = [e10old;e11old];
h1 = E2(:);
Due to the non-linear nature of the function to be optimized, different values of $k_{init}$ should be used to optimize the analysis filter's gain response. The gain responses of the two analysis filters is as shown below. From the gain responses, the minimum stopband attenuation of the analysis filters is observed to be about 23 dB.
M14.3 The MATLAB program used to generate the prototype lowpass filter and the analysis filters of the 4-channel uniform DFT filter bank is given below:

```matlab
L = 19; f = [0 0.15 0.35 1]; m = [1 1 0 0]; w = [10 1];
N = 4; WN = exp(-2*pi*j/N);
plottag = [\'- ';'--';\'-.';': '];
h = zeros(N,L);
n = 0:L-1;
h(1,:) = remez(L-1, f, m, w);
for i = 1:N-1
    h(i+1,:) = h(1,:).*(WN.^(-i*n));
end;
clf;
for i = 1:N
    [H,w] = freqz(h(i,:), 1, 256, 'whole');
    plot(w/pi, abs(H), plottag(i,:));
hold on;
end;
grid on;
hold off;
xlabel('\omega/\pi');ylabel('Magnitude');
title('Magnitude responses of the analysis filter bank');
```

The plots generated by the above program is given below:

M14.4 The first 8 impulse response coefficients of Johnston's 16A lowpass filter $H_L(z)$ are given by

$0.001050167, -0.005054526, -0.002589756, 0.0276414, -0.009666376, -0.09039223, 0.09779817, 0.4810284$

The remaining 8 coefficients are given by flipping the coefficients left to right. From Eq. (14.98), the highpass filter in the tree-structured 3-channel filter bank is given by

$H_2(z) = H_H(z) = z^{-15}H_L(z^{-1})$. The two remaining filters are given by

$H_0(z) = H_L(z)H_L(z^2)$ and $H_1(z) = H_L(z)H_H(z^2)$. The MATLAB program used to generate the gain plots of the 3 analysis filters is given by:

Not for sale 575
G1 = [0.10501670e-2 -0.50545260e-2 -0.25897560e-2
 0.27641400e-1 -0.96663760e-2 -0.90392230e-1 0.97798170e-1
 0.48102840];
G = [G1 flip(G1)];
n = 0:15;
H0 = (-1).^n.*G;
Hsqar = zeros(1,31); Gsqar = zeros(1,31);
Hsqar(1:2:31) = H0; Gsqar(1:2:31) = G;
H1 = conv(Hsqar,G); H2 = conv(Gsqar,G);
[h0,w0] = freqz(H0,[1]); [h1,w1] = freqz(H1,[1]); [h2,w2] =
freqz(H2,[1]);
g0 = 20*log10(abs(h0));g1 = 20*log10(abs(h1));
g2 = 20*log10(abs(h2));
plot(w0/pi,g0,'b-',w1/pi,g1,'r-',w2/pi,g2,'g-.');
axis([0 1 -120 20]);
grid on;
xlabel('\omega/\pi');ylabel('Gain, dB');

The plots generated are given below:

M14.5